

HOMOGENEOUS AFFINE SURFACES: MODULI SPACES

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ABSTRACT. We analyze the moduli space of non-flat homogeneous affine connections on surfaces. For Type \mathcal{A} surfaces, we write down complete sets of invariants that determine the local isomorphism type depending on the rank of the Ricci tensor and examine the structure of the associated moduli space. For Type \mathcal{B} surfaces which are not Type \mathcal{A} we show the corresponding moduli space is a simply connected real analytic 4-dimensional manifold with second Betti number equal to 1.

1. INTRODUCTION

Moduli spaces, representing the space of solutions of geometric classifications, are central in understanding the corresponding set of objects. If the property under consideration is invariant by certain transformations, then the associated moduli space should reflect the action of the corresponding group of transformations. Moduli spaces have many applications both in mathematics and physics and are closely linked to the construction of invariants.

A manifold M equipped with a given structure (affine connection, Riemannian metric, Kähler structure, etc.) is *locally homogeneous* if, given any pair of points of M , there is a germ of a diffeomorphism (isomorphism, isometry, complex isometry, etc.) taking one point into another and preserving the given structure.

In this paper we shall study the moduli spaces of homogeneous affine surfaces. We shall identify two such surfaces if they are affine equivalent for some (orientation preserving) affine transformation. Hence moduli spaces of homogeneous affine surfaces may be interpreted as providing a universal space of parameters to the problem of finding all homogeneous affine connections that may carry some natural geometric and topological structures. The classification of non-flat locally homogeneous affine surfaces was obtained by Opozda [14] (see also [1, 6, 8, 9] for related work). They fall into three non-disjoint families: *Type \mathcal{A}* , *Type \mathcal{B}* and *Type \mathcal{C}* that we shall discuss in more detail in Theorem 1.1. The Type \mathcal{C} surfaces correspond to locally homogeneous affine surfaces where the connection in question is the Levi-Civita connection of a metric with constant Gauss curvature. There are a finite number of such examples depending on the sign of the Gauss curvature and the signature of the underlying manifold. Thus we shall restrict our attention to the study of the moduli spaces of Type \mathcal{A} and Type \mathcal{B} surfaces.

A slightly weaker definition of homogeneity was considered in [7]: a manifold is called *quasihomogeneous* if it is locally homogeneous on a nontrivial open set, but not on the whole surface. There, a classification of torsion-free real-analytic quasihomogeneous affine connections on compact orientable surfaces was given.

We adopt the following notational conventions. An affine surface is a pair $\mathcal{M} = (M, \nabla)$ where ∇ is a torsion free connection on the tangent bundle of a 2-dimensional manifold M . If $x = (x^1, x^2)$ is a system of local coordinates on M , we adopt the *Einstein convention* and sum over repeated indices to define the

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Christoffel symbols $\Gamma = \Gamma_{ij}{}^k$ by expanding:

$$\nabla_{\partial_{x^i}} \partial_{x^j} = \Gamma_{ij}{}^k \partial_{x^k};$$

the condition that ∇ is torsion free is equivalent to the symmetry $\Gamma_{ij}{}^k = \Gamma_{ji}{}^k$ of the Christoffel symbols. Let $R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$ be the curvature operator and let $\rho(X, Y) := \text{Tr}\{Z \rightarrow R(Z, X)Y\}$ be the Ricci tensor.

Affine connections on surfaces have been used to construct new examples of pseudo-Riemannian metrics exhibiting properties without Riemannian counterpart [3, 4, 5, 12]. Flat connections play a distinguished role in many problems and the corresponding moduli spaces, together with their geometric structure, have been broadly investigated in the literature.

Recall that \mathcal{M} is *locally homogeneous* if given any pair of points of M , there is a germ of a diffeomorphism taking one point into another and preserving ∇ . Suppose ∇ is a connection on $M = \mathbb{R}^2$ so that the Christoffel symbols are constant. The translation group $(x^1, x^2) \rightarrow (x^1 + b^1, x^2 + b^2)$ acts transitively on \mathbb{R}^2 and preserves ∇ , so this geometry is homogeneous. Similarly, if ∇ is a connection on $M = \mathbb{R}^+ \times \mathbb{R}$ so that the Christoffel symbols take the form $\Gamma_{ij}{}^k = (x^1)^{-1} C_{ij}{}^k$ for $C_{ij}{}^k$ constant, then the “ $ax + b$ ” group of transformations $(x^1, x^2) \rightarrow (ax^1, ax^2 + b)$ for $a > 0$ and $b \in \mathbb{R}$ acts transitively on M and preserves ∇ so this geometry is homogeneous as well. Finally, the Levi-Civita connection of a simply connected complete Riemann surface of constant non-zero sectional curvature is homogeneous. The following classification result of Opozda [14] shows that these are the only possible geometric models for a locally homogeneous affine surface:

Theorem 1.1. *Let $\mathcal{M} = (M, \nabla)$ be a locally homogeneous affine surface which is not flat. Then at least one of the following three possibilities holds which describe the local geometry:*

- A) *There exists a coordinate atlas so the Christoffel symbols $\Gamma_{ij}{}^k$ are constant.*
- B) *There exists a coordinate atlas so the Christoffel symbols have the form $\Gamma_{ij}{}^k = (x^1)^{-1} C_{ij}{}^k$ for $C_{ij}{}^k$ constant and $x^1 > 0$.*
- C) *∇ is the Levi-Civita connection of a metric of constant sectional curvature.*

We assume \mathcal{M} is not flat which for a surface means that the Ricci tensor ρ does not vanish. One says that \mathcal{M} is *Type A*, *Type B* or *Type C* depending on which possibility holds in Theorem 1.1. These are not exclusive. Although there are no surfaces which are both Type A and Type C, there are surfaces which are both Type A and Type B and there are surfaces which are both Type B and Type C. The Ricci tensor is symmetric for Type A and Type C surfaces but need not be symmetric for Type B surfaces in general. Let \mathfrak{Z}_A (resp. \mathfrak{Z}_B) be the moduli space of isomorphism classes of germs of affine surfaces of Type A (resp. Type B).

In previous work, Kowalski and Vlasek [11] showed that the moduli space of Type A connections is at most two-dimensional and that the moduli space of Type B connections is at most four dimensional. Our purpose is to understand these moduli spaces in more detail. We shall investigate their underlying structure and construct some new invariants which are not of Weyl type. For Type A surfaces, the invariants we shall construct completely characterize the homogeneous affine structure up to affine equivalence. We now summarize our main results.

1.1. Type A surfaces. Let \mathcal{Z}_A be the set of Christoffel symbols $\Gamma \in \mathbb{R}^6$ defining a Type A structure. Thus, let

$$\Gamma := (\Gamma_{11}{}^1, \Gamma_{11}{}^2, \Gamma_{12}{}^1 = \Gamma_{21}{}^1, \Gamma_{12}{}^2 = \Gamma_{21}{}^2, \Gamma_{22}{}^1, \Gamma_{22}{}^2) \in \mathbb{R}^6$$

determine a translation invariant homogeneous affine structure on \mathbb{R}^2 . The Ricci tensor is given by

$$\begin{aligned}\rho_{11} &= \Gamma_{12}^2(\Gamma_{11}^1 - \Gamma_{12}^2) + \Gamma_{11}^2(\Gamma_{22}^2 - \Gamma_{12}^1), \\ \rho_{12} &= \rho_{21} = \Gamma_{12}^1\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^1, \\ \rho_{22} &= \Gamma_{22}^1(\Gamma_{11}^1 - \Gamma_{12}^2) + \Gamma_{12}^1(\Gamma_{22}^2 - \Gamma_{12}^1).\end{aligned}\tag{1.a}$$

Thus, in particular, ρ is symmetric if \mathcal{M} is of Type \mathcal{A} . If $\rho = 0$, then \mathcal{M} is flat. As we shall assume \mathcal{M} is not flat, either $\text{Rank}\{\rho\} = 1$ or $\text{Rank}\{\rho\} = 2$.

1.1.1. *Type \mathcal{A} surfaces with $\text{Rank}\{\rho\} = 1$.* Let $\mathcal{M} = (M, \nabla)$ be an affine surface of Type \mathcal{A} with $\text{Rank}\{\rho\} = 1$. We showed previously [2] that \mathcal{M} is both of Type \mathcal{A} and of Type \mathcal{B} implies $\text{Rank}\{\rho\} = 1$. Let P be an arbitrary point of M ; the particular point is irrelevant since \mathcal{M} is locally homogeneous. Choose $X \in T_P M$ so that $\rho(X, X) \neq 0$. Let $\nabla\rho$ be the covariant derivative of the Ricci tensor. Define

$$\alpha_X(\mathcal{M}) := \nabla\rho(X, X; X)^2 \cdot \rho(X, X)^{-3} \text{ and } \epsilon_X(\mathcal{M}) := \text{sign}(\rho(X, X)) = \pm 1.$$

The following result [2] shows that α_X and ϵ_X give a complete system of invariants for such surfaces:

Theorem 1.2. *Let \mathcal{M} be an affine surface of Type \mathcal{A} with $\text{Rank}\{\rho\} = 1$. Then α_X and ϵ_X are independent of X and define affine invariants $\alpha(\mathcal{M})$ and $\epsilon(\mathcal{M})$.*

- (1) \mathcal{M} is symmetric (i.e. $\nabla\rho = 0$) if and only if $\alpha(\mathcal{M}) = 0$.
- (2) If $\alpha(\mathcal{M}) \neq 0$, then \mathcal{M} is also of Type \mathcal{B} if and only if $\alpha(\mathcal{M}) \notin (0, 16)$.
- (3) If $\alpha(\mathcal{M}) = 0$, then \mathcal{M} is also of Type \mathcal{B} if and only if $\epsilon < 0$.
- (4) If $\tilde{\mathcal{M}}$ is another Type \mathcal{A} surface with $\text{Rank}(\tilde{\rho}) = 1$, with $\alpha(\tilde{\mathcal{M}}) = \alpha(\mathcal{M})$, and with $\epsilon(\tilde{\mathcal{M}}) = \epsilon(\mathcal{M})$, then $\tilde{\mathcal{M}}$ is locally isomorphic to \mathcal{M} .

1.1.2. *Type \mathcal{A} surfaces with $\text{Rank}\{\rho\} = 2$.* If \mathcal{M} is a surface of Type \mathcal{A} with $\text{Rank}\{\rho\} = 2$, then \mathcal{M} is not of Type \mathcal{B} . The structure group is the affine group

$$\begin{aligned}\mathfrak{F} &:= \{T \in \text{Diff}(\mathbb{R}^2) : T(x^1, x^2) = (a_1^1 x^1 + a_2^1 x^2 + b^1, a_1^2 x^1 + a_2^2 x^2 + b^2) \\ &\text{where } (a_i^j) \in \text{GL}(2, \mathbb{R})\}.\end{aligned}$$

If $T \in \mathfrak{F}$ then $T_* \partial_{x^i} = \tilde{a}_j^i \partial_{x^j}$, where \tilde{a}_j^i are the components of the inverse matrix so $\sum_j a_i^j \tilde{a}_j^k = \delta_i^k$ is the Kronecker symbol. If $\Gamma \in \mathbb{R}^6$, let $T^* \Gamma$ be the associated Christoffel symbols in the new coordinate system; these are constant and given in the form:

$$(T^* \Gamma)_{ij}^k = \sum_{\alpha, \beta, \gamma} \tilde{a}_i^\alpha \tilde{a}_j^\beta a_\gamma^k \Gamma_{\alpha\beta}^\gamma$$

The translations play no role and we obtain a representation of $\text{GL}(2, \mathbb{R})$ on \mathbb{R}^6 . Fix a Type \mathcal{A} coordinate atlas $\{(\mathcal{O}_\alpha, \phi_\alpha)\}$ on M . Here $\{\mathcal{O}_\alpha\}$ forms an open cover of M so the Christoffel symbols are constant and the maps ϕ_α are diffeomorphisms from \mathcal{O}_α to an open subset of \mathbb{R}^2 . Sum over repeated indices to define:

$$\rho_{ij}^3 := \Gamma_{ik}^l \Gamma_{jl}^k, \quad \psi_3 := \text{Tr}_\rho\{\rho^3\} = \rho^{ij} \rho_{ij}^3, \quad \Psi_3 := \det(\rho^3) / \det(\rho). \tag{1.b}$$

The transition functions of a Type \mathcal{A} coordinate atlas on \mathcal{M} belong to the affine group \mathfrak{F} , and ψ_3 and Ψ_3 are independent of the local Type \mathcal{A} coordinates chosen and thus are affine invariants of \mathcal{M} (see Lemma 2.1). We can use the local homogeneity to choose the coordinate atlas so that ${}^\alpha \Gamma = \Gamma$ is independent of the particular chart \mathcal{O}_α . Let $\rho = \rho_\Gamma$; this is a fixed non-degenerate bilinear form. The transition functions then belong to the orthogonal affine group $\mathfrak{F}^\rho := \{T \in \mathfrak{F} : T^* \rho = \rho\}$, i.e. (a_{ij}) is in the orthogonal group determined by ρ .

Definition 1.3. Let \mathcal{Z}_+ (resp. \mathcal{Z}_0 or \mathcal{Z}_-) be the set of Christoffel symbols $\Gamma \in \mathbb{R}^6$ defining a Type \mathcal{A} structure such that the Ricci tensor is positive definite (resp. indefinite or negative definite). Let \mathfrak{Z}_ε for $\varepsilon = +, 0, -$ be the associated moduli space.

It will follow from Lemma 2.1 below that $\mathfrak{Z}_\varepsilon = \mathcal{Z}_\varepsilon / \text{GL}(2, \mathbb{R})$. Since \mathcal{Z}_ε is an open subset of \mathbb{R}^6 and since $\text{GL}(2, \mathbb{R})$ is a 4-dimensional Lie group, one expects that \mathfrak{Z}_ε will be 2-dimensional. Kowalski and Vlasek [11] have shown that this is the case.

Let $\Theta_\varepsilon := (\psi_3, \Psi_3)$ on \mathcal{Z}_ε ; this real analytic map extends to a map from the moduli space \mathfrak{Z}_ε to \mathbb{R}^2 . The following result makes this very explicit and gives a complete set of real analytic invariants:

Theorem 1.4. Θ_ε is a 1-1 map from \mathfrak{Z}_ε to a closed simply connected subset of \mathbb{R}^2 .

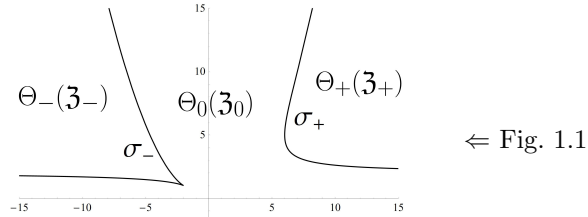
We discuss the range of Θ_ε as follows. Consider the two curves

$$\begin{aligned}\sigma_+(t) &:= (4t^2 + \frac{1}{t^2} + 2, 4t^4 + 4t^2 + 2), \\ \sigma_-(t) &:= (-4t^2 - \frac{1}{t^2} + 2, 4t^4 - 4t^2 + 2).\end{aligned}\tag{1.c}$$

The curve σ_+ is smooth; the curve σ_- has a cusp at $(-2, 1)$ when $t = \frac{1}{\sqrt{2}}$. These two curves divide the plane into 3 open regions \mathfrak{D}_- , \mathfrak{D}_0 , and \mathfrak{D}_+ where \mathfrak{D}_- lies in the second quadrant and is bounded on the right by σ_- , \mathfrak{D}_+ lies in the first quadrant, and is bounded on the left by σ_+ and \mathfrak{D}_0 lies in between and is bounded on the left by σ_- and on the right by σ_+ . Let \mathfrak{C}_ε be the closure of \mathfrak{D}_ε ; $\mathfrak{C}_- = \mathfrak{D}_- \cup \text{range}(\sigma_-)$, $\mathfrak{C}_+ = \mathfrak{D}_+ \cup \text{range}(\sigma_+)$, and $\mathfrak{C}_0 = \mathfrak{D}_0 \cup \text{range}(\sigma_-) \cup \text{range}(\sigma_+)$.

Theorem 1.5. Adopt the notation established above. Then $\Theta_\varepsilon(\mathcal{Z}_\varepsilon) = \mathfrak{C}_\varepsilon$.

In Fig. 1.1 below, we show the two curves σ_\pm which bound the images of the moduli spaces $\Theta_-(\mathfrak{Z}_-)$, $\Theta_0(\mathfrak{Z}_0)$, and $\Theta_+(\mathfrak{Z}_+)$



← Fig. 1.1

Note that although Θ_ε is 1-1 on \mathfrak{Z}_ε , we have that $\Theta_+(\mathfrak{Z}_+)$ intersects $\Theta_0(\mathfrak{Z}_0)$ along their common boundary σ_+ and that $\Theta_-(\mathfrak{Z}_-)$ intersects $\Theta_0(\mathfrak{Z}_0)$ along their common boundary σ_- . This does not mean that \mathfrak{Z}_+ (resp. \mathfrak{Z}_-) intersects \mathfrak{Z}_0 nor does it mean that Θ_- , or Θ_0 , or Θ_+ are not 1-1 on their respective domains.

We have worked in the category of unoriented surfaces to set $\mathfrak{Z}_\varepsilon = \mathcal{Z}_\varepsilon / \text{GL}(2, \mathbb{R})$. If we work in the category of oriented surfaces, we may obtain a corresponding moduli space $\mathfrak{Z}_\varepsilon^+ = \mathcal{Z}_\varepsilon / \text{GL}^+(2, \mathbb{R})$ of oriented affine surfaces of Type \mathcal{A} where $\text{GL}^+(2, \mathbb{R})$ denotes the group of matrices with positive determinant. If $\Gamma \in \mathcal{Z}_\varepsilon$, let

$$\begin{aligned}\mathcal{G}^+(\Gamma) &:= \{T \in \text{GL}^+(2, \mathbb{R}) : T^*\Gamma = \Gamma\}, \\ \mathcal{G}(\Gamma) &:= \{T \in \text{GL}(2, \mathbb{R}) : T^*\Gamma = \Gamma\}\end{aligned}$$

be the isotropy subgroups of the actions. Let $\text{dvol} := |\det(\rho_{ij})|^{1/2} dx^1 \wedge dx^2$ be the oriented volume form. Extend ρ to an inner product on tensors of all type and sum over repeated indices to define an invariant χ which is sensitive to the orientation:

$$\chi(\Gamma) := \rho(\Gamma_{ab}{}^b \Gamma_{ij}{}^k \rho_{kl}^3 \rho^{ij} dx^a \wedge dx^l, \text{dvol}). \tag{1.d}$$

Since this is given in terms of contraction of indices, it is an affine invariant by Lemma 2.1.

The cusp point $\Gamma_{\text{csp}} \in \mathfrak{Z}_-$ which satisfies $\Theta_-(\Gamma_{\text{csp}}) = (-2, 1)$ is singular and corresponds to the structure $\sigma_-(\frac{1}{\sqrt{2}})$;

$$[\Gamma_{\text{csp}}] = \{[\Gamma_{11}^1 = -\frac{1}{\sqrt{2}}, \Gamma_{11}^2 = 0, \Gamma_{12}^1 = 0, \Gamma_{12}^2 = \frac{1}{\sqrt{2}}, \Gamma_{22}^1 = \frac{1}{\sqrt{2}}, \Gamma_{22}^2 = 0]\}.$$

Theorem 1.6 shows the sequence $\text{GL}^+(2, \mathbb{R}) \rightarrow \mathcal{Z}_\varepsilon \rightarrow \mathfrak{Z}_\varepsilon^+$ is a principle bundle except at the cusp point of \mathfrak{C}_- . Furthermore, a structure Γ with a given orientation is isomorphic to Γ with the opposite orientation if and only if Γ belongs to the boundary of one of the regions. This shows that $\mathfrak{Z}_\varepsilon^+$ is the double of \mathfrak{Z}_ε where we glue two copies of \mathfrak{Z}_ε together along the boundary as described above in Fig. 1.1.

Theorem 1.6.

- (1) Let $\Gamma \in \mathcal{Z}_+$. Then $|\mathcal{G}^+(\Gamma)| = 1$.
 - (a) If $\Theta_+\Gamma \in \text{bd}(\mathfrak{C}_+)$, then $|\mathcal{G}(\Gamma)| = 2$.
 - (b) If $\Theta_+\Gamma \notin \text{bd}(\mathfrak{C}_+)$, then $|\mathcal{G}(\Gamma)| = 1$.
- (2) Let $\Gamma \in \mathcal{Z}_-$.
 - (a) If $\Theta_-(\Gamma) = (-2, 1)$, then $|\mathcal{G}^+(\Gamma)| = 3$ and $|\mathcal{G}(\Gamma)| = 6$.
 - (b) If $\Theta_-(\Gamma) \in \text{bd}(\mathfrak{C}_+) - \{(-2, 1)\}$, then $|\mathcal{G}^+(\Gamma)| = 1$ and $|\mathcal{G}(\Gamma)| = 2$.
 - (c) If $\Theta_-(\Gamma) \notin \text{bd}(\mathfrak{C}_+)$, then $|\mathcal{G}^+(\Gamma)| = 1$ and $|\mathcal{G}(\Gamma)| = 1$.
- (3) Let $\Gamma \in \mathcal{Z}_0$. Then $|\mathcal{G}^+(\Gamma)| = 1$.
 - (a) If $\Theta_0\Gamma \in \text{bd}(\mathfrak{C}_0)$, then $|\mathcal{G}(\Gamma)| = 2$.
 - (b) If $\Theta_0\Gamma \notin \text{bd}(\mathfrak{C}_0)$, then $|\mathcal{G}(\Gamma)| = 1$.
- (4) The map $\tilde{\Theta}_\varepsilon := (\psi_3, \Psi_3, \chi)$ is a 1–1 embedding of \mathfrak{Z}_ε into \mathbb{R}^3 for $\varepsilon = -, 0, +$. The embedding Θ_ε is the projection of $\tilde{\Theta}_\varepsilon$ on the horizontal plane. It is a “fold” map that is 1–1 except along the boundary of $\Theta_\varepsilon(\mathfrak{Z}_\varepsilon)$ which is locus where $\chi = 0$.

1.2. Type \mathcal{B} surfaces. We now examine the moduli space of Type \mathcal{B} surfaces $\mathfrak{Z}_\mathcal{B}$. Let $\mathcal{M}_C := (\mathbb{R}^+ \times \mathbb{R}, \Gamma = (x^1)^{-1}C)$ be the surface of Type \mathcal{B} determined by $C \in \mathbb{R}^6$. Let $\kappa(\mathcal{M}_C)$ be the dimension of the space of affine Killing vector fields on \mathcal{M}_C . We established the following result previously [2]:

Theorem 1.7. *If \mathcal{M}_C is not flat, then $2 \leq \kappa \leq 4$. Furthermore, \mathcal{M} is also Type \mathcal{A} if and only if $\kappa = 4$.*

Since the case that \mathcal{M}_C is also of Type \mathcal{A} has already been dealt with, we shall assume henceforth that $2 \leq \kappa \leq 3$. Let

$$\mathcal{Z}_{23\mathcal{B}} := \{C \in \mathbb{R}^6 : 2 \leq \kappa(\mathcal{M}_C) \leq 3\} \subset \mathbb{R}^6$$

and let $\mathfrak{Z}_{23\mathcal{B}}$ be the associated moduli space.

Let $C \in \mathcal{Z}_{23\mathcal{B}}$ define a homogeneous structure on $\mathbb{R}^+ \times \mathbb{R}$ with $2 \leq \kappa(\mathcal{M}_C) \leq 3$. Use $\pm dx^1 \wedge dx^2$ to define the corresponding oriented surface \mathcal{M}_C^\pm . Let $\mathfrak{Z}_{23\mathcal{B}}^+$ be the associated moduli space of oriented surfaces with $2 \leq \kappa(\mathcal{M}_C^\pm) \leq 3$. Let

$$\begin{aligned} \mathfrak{G} &:= \{T : T(x^1, x^2) = (ax^1, bx^1 + cx^2 + d) \text{ for } a > 0 \text{ and } c \neq 0\}, \\ \mathfrak{J} &:= \{T_{b,c} \in \mathfrak{G} : T_{b,c}(x^1, x^2) = (x^1, bx^1 + cx^2) \text{ for } c \neq 0\}, \\ \mathfrak{J}^+ &:= \{T_{b,c} \in \mathfrak{J} \text{ for } c > 0\}. \end{aligned}$$

Our previous discussion in [2] permits us to identify

$$\mathfrak{Z}_{23\mathcal{B}} = \mathcal{Z}_{23\mathcal{B}}/\mathfrak{J} \text{ and } \mathfrak{Z}_{23\mathcal{B}}^+ = \mathcal{Z}_{23\mathcal{B}}/\mathfrak{J}^+.$$

This is a non-trivial assertion if $\kappa(\mathcal{M}) = 3$ as there are non-linear affine transformations. However, they play no role in defining the affine isomorphism type. We

define several invariant tensors on $\mathfrak{Z}_{23\mathcal{B}}$ that allow us to introduce coordinates and to obtain the structure of the moduli space $\mathfrak{Z}_{23\mathcal{B}}$:

$$\begin{aligned}\rho_1 &:= \frac{1}{x^1} \{ \Gamma_{12}^2 dx^1 \otimes dx^1 + \Gamma_{22}^2 dx^1 \otimes dx^2 - \Gamma_{12}^1 dx^2 \otimes dx^1 - \Gamma_{22}^1 dx^2 \otimes dx^2 \}, \\ \rho_2 &:= \Gamma_{ij}^k \Gamma_{kl}^l dx^i \otimes dx^j, \quad \rho_3 := \Gamma_{ik}^l \Gamma_{jl}^k dx^i \otimes dx^j, \quad \rho_0 := \Gamma_{ij}^j dx^i, \\ \rho_4 &:= \Gamma_{ij}^k C_{ak}^j C_{bc}^i dx^a \otimes dx^b \otimes dx^c.\end{aligned}$$

We shall show in Lemma 5.1 that these tensors are invariantly defined.

In Lemma 5.2, we will examine the isotropy subgroup of the natural action of \mathfrak{J}^+ on $\mathcal{K}_{23\mathcal{B}}$, we will show that $\mathcal{K}_{23\mathcal{B}}$ is a 4-dimensional real analytic manifold, and we will prove the natural projection π^+ from $\mathcal{K}_{23\mathcal{B}}$ to $\mathfrak{R}_{23\mathcal{B}}^+$ is a real-analytic \mathfrak{J}^+ fiber bundle. In Lemma 5.4, we show that $\mathfrak{Z}_{23\mathcal{B}}^+$ is simply connected and that the second Betti number is 2. Our main result concerning Type \mathcal{B} structures is summarized in the following:

Theorem 1.8. *The moduli space $\mathfrak{Z}_{23\mathcal{B}}$ is a 4-dimensional real analytic manifold, the natural projection from $\mathcal{Z}_{23\mathcal{B}}$ to $\mathfrak{Z}_{23\mathcal{B}}$ is real analytic, $\mathfrak{Z}_{23\mathcal{B}}$ is simply connected, and the second Betti number of $\mathfrak{Z}_{23\mathcal{B}}$ is 1.*

In Theorem 5.5 we complete the proof of Theorem 1.8 by giving $\mathfrak{Z}_{23\mathcal{B}}$ a real analytic structure, by showing that the projection from $\mathfrak{Z}_{23\mathcal{B}}^+$ to $\mathfrak{Z}_{23\mathcal{B}}$ is a \mathbb{Z}_2 branched cover where the ramification set is a real analytic sub-manifold of co-dimension 2, and by demonstrating that $\mathfrak{Z}_{23\mathcal{B}}^+$ is simply connected and has second Betti number equal to 1.

1.3. Outline of the paper. In Section 2, we prove Theorem 1.4 in the positive and negative definite settings. In Section 3, we prove Theorem 1.4 in the indefinite setting. In Section 4 we prove Theorem 1.5 and Theorem 1.6 and in Section 5 we prove Theorem 1.8.

2. THE MODULI SPACES \mathfrak{Z}_{\pm}

Recall from [2] that if \mathcal{M} is a surface of Type \mathcal{A} with $\text{Rank}\{\rho\} = 2$, then \mathcal{M} is not of Type \mathcal{B} . Moreover the structure group is the affine group

$$\begin{aligned}\mathfrak{F} &:= \{T \in \text{Diff}(\mathbb{R}^2) : T(x^1, x^2) = (a_1^1 x^1 + a_2^1 x^2 + b^1, a_1^2 x^1 + a_2^2 x^2 + b^2) \\ &\quad \text{where } (a_i^j) \in \text{GL}(2, \mathbb{R})\}.\end{aligned}$$

Further observe that since any Type \mathcal{A} surface is projectively flat (see [3]), if $\text{Rank}\{\rho\} = 2$, then the Ricci tensor defines a metric so that (ρ, ∇) is a Codazzi pair, also called a statistical structure (see [13] and references therein).

Lemma 2.1. *Let $\mathcal{M} = (M, \nabla)$ be a Type \mathcal{A} affine surface with $\text{Rank}(\rho) = 2$. Then transition functions of a Type \mathcal{A} coordinate atlas on \mathcal{M} belong to the affine group \mathfrak{F} , and ψ_3 and Ψ_3 defined in Equation (1.b) are independent of the local Type \mathcal{A} coordinates chosen and thus are affine invariants of \mathcal{M} .*

Proof. Cover M by Type \mathcal{A} coordinate charts $(\mathcal{O}_\alpha, \phi_\alpha)$ so ${}^\alpha\Gamma$ is constant. The transition functions $\phi_{\alpha\beta}$ then are local diffeomorphisms of \mathbb{R}^2 so that $\phi_{\alpha\beta}^* \{\beta\rho\} = {}^\alpha\rho$. The Ricci tensors ${}^\alpha\rho$ and ${}^\beta\rho$ define flat pseudo-Riemannian metrics. This implies $d\phi_{\alpha\beta}$ is constant and, consequently, $\phi_{\alpha\beta}$ is an affine linear transformation. Since contracting upper and lower indices is invariant under the action of $\text{GL}(2, \mathbb{R})$, ψ_3 and Ψ_3 are invariants. \square

We begin by parametrizing the moduli spaces \mathfrak{Z}_{\pm} .

Definition 2.2. Let $\mathcal{S} := (0, \infty) \times [0, \infty)$. If $(x, y) \in \mathbb{R}^2$ with $x \neq 0$, define:

$$\Gamma_{\pm}(x, y) := \{\Gamma_{11}^1 = x \pm \frac{1}{x}, \Gamma_{11}^2 = 0, \Gamma_{12}^1 = 0, \Gamma_{12}^2 = x, \Gamma_{22}^1 = x, \Gamma_{22}^2 = y\}.$$

By Equation (1.a), $\rho(\Gamma_{\pm}(x, y)) = \pm \text{Id}$ so Γ_{\pm} takes values in \mathcal{Z}_{\pm} .

If $\Gamma \in \mathcal{Z}_{\pm}$, let $[\Gamma]$ denote the corresponding element of \mathfrak{Z}_{\pm} .

Lemma 2.3. *If $[\Gamma] \in \mathfrak{Z}_{\pm}$, then there exists $(x, y) \in \mathcal{S}$ so $[\Gamma] = [\Gamma_{\pm}(x, y)]$.*

Proof. Choose $[\Gamma] \in \mathfrak{Z}_{\pm}$. We will show that there exists $T \in \text{GL}(2, \mathbb{R})$ and a point $(x, y) \in \mathcal{S}$ so that $T^*\Gamma = \Gamma_{\pm}(x, y)$, i.e. that $\Gamma = \Gamma_{\pm}(x, y)$ after an appropriate change of coordinates. We begin by making a linear change of coordinates to assume $\rho_{\Gamma} = \pm \text{Id}$. Consider the rotation

$$\begin{aligned} T_{\theta}(\partial_{x^1}) &:= \cos(\theta)\partial_{x^1} + \sin(\theta)\partial_{x^2}, \\ T_{\theta}(\partial_{x^2}) &:= -\sin(\theta)\partial_{x^1} + \cos(\theta)\partial_{x^2}. \end{aligned} \quad (2.a)$$

Let $(T_{\theta}^*\Gamma)$ be the expression of Γ in this new coordinate system. Then

$$\begin{aligned} (T_{\theta}^*\Gamma)_{12}^1 &= \cos^3(\theta)\Gamma_{12}^1 + \cos^2(\theta)\sin(\theta)(\Gamma_{22}^1 + \Gamma_{12}^2 - \Gamma_{11}^1) \\ &\quad + \cos(\theta)\sin^2(\theta)\{\Gamma_{22}^2 - \Gamma_{12}^1 - \Gamma_{11}^2\} - \sin^3(\theta)\Gamma_{12}^2. \end{aligned}$$

Since $(T_{\pi}^*\Gamma)_{12}^1 = -(T_0^*\Gamma)_{12}^1$, the Intermediate Value Theorem shows that there exists θ so $(T_{\theta}^*\Gamma)_{12}^1 = 0$. There may, of course, be other values where this happens and this will play a role in our subsequent development. It is exactly this step which fails in the indefinite setting as the structure group is $O(1, 1)$ and not $O(2)$.

Normalize the coordinate system so that $\rho = \pm \text{Id}$ and $\Gamma_{12}^1 = 0$. Because $\rho_{12} = -\Gamma_{11}^2\Gamma_{22}^1 = 0$ and $\rho_{22} = (\Gamma_{11}^1 - \Gamma_{12}^2)\Gamma_{22}^1 \neq 0$, we conclude $\Gamma_{22}^1 \neq 0$ and $\Gamma_{11}^2 = 0$ so $\rho_{11} = (\Gamma_{11}^1 - \Gamma_{12}^2)\Gamma_{12}^2$ and $\rho_{22} = (\Gamma_{11}^1 - \Gamma_{12}^2)\Gamma_{22}^1$. Set $x := \Gamma_{12}^2 \neq 0$ and $y := \Gamma_{22}^2$; $x \neq 0$. By changing the sign of x^1 and/or x^2 , we may assume $x > 0$ and $y \geq 0$. We have $\Gamma_{22}^1 = x$. We solve the equation $(\Gamma_{11}^1 - x)x = \pm 1$ to determine Γ_{11}^1 and obtain the normalizations defining Γ_{\pm} to complete the proof. \square

We establish some notation to use subsequently. Set

$$p_{\pm}(x, y) := \psi_3(\Gamma_{\pm}(x, y)) \text{ and } P_{\pm}(x, y) := \Psi_3(\Gamma_{\pm}(x, y))$$

to regard the invariants of Lemma 2.1 as functions on \mathcal{S} . A direct computation then shows:

$$\begin{aligned} p_{\pm}(x, y) &:= \pm 4x^2 \pm \frac{1}{x^2} \pm y^2 + 2, \\ P_{\pm}(x, y) &:= 4x^4 + x^2(y^2 \pm 4) + \frac{y^2}{x^2} + 2(1 \pm y^2). \end{aligned} \quad (2.b)$$

Let $\Theta_{\pm} := (p_{\pm}, P_{\pm}) : \mathcal{S} \rightarrow \mathbb{R}^2$ and let $J_{\pm} := \det\{\Theta'_{\pm}\}$ be the Jacobian determinant. A direct computation shows:

$$\begin{aligned} J_{-}(x, y) &:= \frac{4(x^2+1)y(4x^6+x^4y^2-x^2(y^2+3)+1)}{x^5}, \\ J_{+}(x, y) &:= -\frac{4(x^2-1)y(4x^6+x^4y^2+x^2(y^2-3)-1)}{x^5}. \end{aligned} \quad (2.c)$$

Thus Θ_{\pm} is a local diffeomorphism from \mathcal{S} to \mathbb{R}^2 except where $J_{\pm}(x, y) = 0$. Set

$$\begin{aligned} Y_{-}(x) &:= \sqrt{\frac{1+4x^6-3x^2}{x^2-x^4}} \quad \text{for } 0 < x < 1, \\ Y_{+}(x) &:= \sqrt{\frac{-4x^6+3x^2+1}{x^4+x^2}} \quad \text{for } 0 < x \leq 1. \end{aligned} \quad (2.d)$$

Let $L_{\pm} := \{(x, y) \in \mathcal{S} : J_{\pm}(x, y) = 0\}$ be the *Jacobi locus*;

$$\begin{aligned} L_{-} &:= \{(x, y) \in \mathcal{S} : y = Y_{-}(x) \text{ for } 0 < x < 1 \text{ or } y = 0\}, \\ L_{+} &:= \{(x, y) \in \mathcal{S} : y = Y_{+}(x) \text{ for } 0 < x \leq 1 \text{ or } y = 0 \text{ or } x = 1\}. \end{aligned} \quad (2.e)$$

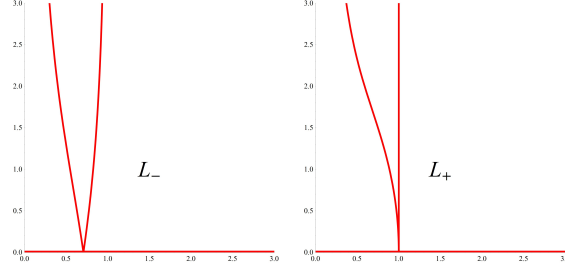
Fig. 2.1 \uparrow Fig. 2.2 \uparrow

Fig. 2.1 gives the Jacobi locus L_- and Fig. 2.2 gives the Jacobi locus L_+ . The x axis is always on the Jacobi locus. The half ray $x = 1$ is in L_+ but not in L_- . The graph $y = Y_+(x)$ touches the x -axis when $x = 1$. The graph $y = Y_-(x)$ touches the x -axis when $x = \frac{1}{\sqrt{2}}$.

2.1. The moduli space \mathfrak{Z}_+ . Adopt the notation of Definition 2.2. We may set $T_1(x^1, x^2) := (-x^1, x^2)$. Then $T_1^* \Gamma_+(x, y) = \Gamma_+(-x, y)$ so $[\Gamma_+(x, y)] = [\Gamma_+(-x, y)]$. Similarly we may use $T_2(x^1, x^2) = (x^1, -x^2)$ to see $[\Gamma_+(x, y)] = [\Gamma_+(x, -y)]$. Consequently, we can change the sign of x and/or of y as desired. By Lemma 2.1, $[\Gamma_+(x, y)] = [\Gamma_+(\bar{x}, \bar{y})]$ if and only if $T^* \Gamma_+(x, y) = \Gamma_+(\bar{x}, \bar{y})$ for some $T \in \text{GL}(2, \mathbb{R})$. Since the Ricci tensor is normalized to $\pm \text{diag}(1, 1)$, this implies $T \in O(2)$; after replacing x^1 by $-x^1$ (or x^2 by $-x^2$) if necessary, we may assume $T \in SO(2)$. This means that there exists a rotation T_θ (see Equation (2.a)) so that

$$T_\theta^* \{\Gamma_+(x, y)\}_{12}^1 = 0, \quad T_\theta^* \{\Gamma_+(x, y)\}_{12}^2 = \pm x, \quad T_\theta^* \{\Gamma_+(x, y)\}_{22}^2 = \pm y.$$

We extend Equation (2.a) to the conformal group. For $(u, v) \neq (0, 0)$, define:

$$T_{u,v}(\partial_{x^1}) = u\partial_{x^1} + v\partial_{x^2}, \quad T_{u,v}(\partial_{x^2}) = -v\partial_{x^1} + u\partial_{x^2}. \quad (2.f)$$

We recover T_θ by setting $\cos(\theta) = \frac{u}{\sqrt{u^2+v^2}}$ and $\sin(\theta) = \frac{v}{\sqrt{u^2+v^2}}$. We examine the equation $\{T^* \Gamma_+(x, y)\}_{12}^1 = 0$. This yields the homogeneous cubic equation

$$v\{u^2(x^2 - 1) + uvxy - v^2x^2\} = 0.$$

If $v = 0$, then $T_{1,0}$ is the identity map. Thus we may assume $v \neq 0$. Since we may replace (u, v) by $(-u, -v)$, we may assume $v > 0$. We normalize to assume $v = 1$ and obtain the quadratic equation:

$$u^2(x^2 - 1) + uxy - x^2 = 0. \quad (2.g)$$

Equation (2.g) has at most 2 solutions $u_\pm(x, y)$; we shall set $u_\pm = \text{DNE}$ if no solution exists. Assume $x > 0$ and set:

$$u_\pm(x, y) := \begin{cases} \text{DNE} & \text{if } x = 1 \text{ and } y = 0 \\ \text{DNE} & \text{if } x \neq 1 \text{ and } y^2 + 4(x^2 - 1) < 0 \\ \frac{-y}{2(x^2 - 1)}x & \text{if } y^2 + 4(x^2 - 1) = 0 \\ \frac{x}{y} & \text{if } x = 1 \text{ and } y > 0 \\ \frac{-y \pm \sqrt{y^2 + 4(x^2 - 1)}}{2(x^2 - 1)}x & \text{if } x \neq 1 \text{ and } y^2 + 4(x^2 - 1) > 0 \end{cases}. \quad (2.h)$$

Assuming u_\pm is well defined, the corresponding rotation angles θ_\pm and solutions $(x_\pm(x, y), y_\pm(x, y))$ to the equation $[\Gamma(x, y)] = [\Gamma(x_\pm, y_\pm)]$ are given by:

$$\begin{aligned} \cos_\pm(x, y) &:= \frac{u_\pm(x, y)}{\sqrt{1 + u_\pm(x, y)^2}}, \quad \sin_\pm(x, y) := \frac{1}{\sqrt{1 + u_\pm(x, y)^2}}, \\ x_\pm(x, y) &:= T_{\cos_\pm(x, y), \sin_\pm(x, y)}^* \{\Gamma_+(x, y)\}_{22}^1, \\ y_\pm(x, y) &:= T_{\cos_\pm(x, y), \sin_\pm(x, y)}^* \{\Gamma_+(x, y)\}_{22}^2. \end{aligned}$$

Of course x_\pm and y_\pm can be negative, so we must take the absolute values. This yields the following result:

Lemma 2.4. *Let $(x, y) \in \mathcal{S}$ and $(\bar{x}, \bar{y}) \in \mathcal{S}$. Then $[\Gamma_+(x, y)] = [\Gamma_+(\bar{x}, \bar{y})]$ if and only if at least one of the following 3 conditions (which need not be exclusive) holds:*

- (1) $(\bar{x}, \bar{y}) = (x, y)$.
- (2) $x_-(x, y)$ and $y_-(x, y)$ are defined and $(\bar{x}, \bar{y}) = (|x_-(x, y)|, |y_-(x, y)|)$.
- (3) $x_+(x, y)$ and $y_+(x, y)$ are defined and $(\bar{x}, \bar{y}) = (|x_+(x, y)|, |y_+(x, y)|)$.

This shows that the map $(x, y) \rightarrow [\Gamma_+(x, y)]$ is at most a $3 \rightarrow 1$ map from \mathcal{S} to \mathfrak{Z}_+ . If $(x, y) \in \mathcal{S}$, let $1 \leq n_+(x, y) \leq 3$ be the number of distinct points $(\bar{x}, \bar{y}) \in \mathcal{S}$ which satisfy $[\Gamma_+(x, y)] = [\Gamma_+(\bar{x}, \bar{y})]$. To determine $n_+(x, y)$, we examine the 5 possibilities of Equation (2.h).

Case 2.1.1. Let $(x, y) = (1, 0)$. As Equation (2.g) has no solutions, $n_+(x, y) = 1$.

Case 2.1.2. Let $y^2 + 4(x^2 - 1) < 0$. As Equation (2.g) has no solutions, $n_+(x, y) = 1$.

Case 2.1.3. Let $y^2 + 4(x^2 - 1) = 0$ and $0 < x < 1$. Let

$$\begin{aligned}\mathcal{D} &:= \{(x, y) \in \mathcal{S} : y^2 + 4(x^2 - 1) = 0\}, \\ \mathcal{R} &:= \{(x, y) \in \mathcal{S} : x = 1 \text{ and } y > 0\}\end{aligned}$$

be the *discriminant locus* and a ray in \mathcal{S} , respectively. We define \mathcal{D} and \mathcal{R} by setting, respectively: $P_{\mathcal{D}}(t) := (t, 2\sqrt{1-t^2})$ and $P_{\mathcal{R}}(t) := (1, \frac{\sqrt{1-t^2}}{t})$ for $t \in (0, 1)$. If $(x, y) = P_{\mathcal{D}}(t) \in \mathcal{D}$, then there is only one solution $(x_{\pm}, y_{\pm}) = (1, -\frac{\sqrt{1-t^2}}{t})$. Thus after changing the sign of y_{\pm} , we see $[\Gamma_+(P_{\mathcal{D}}(t))] = [\Gamma_+(P_{\mathcal{R}}(t))]$ and $n = 2$ on \mathcal{D} and on \mathcal{R} .

Case 2.1.4. Let $(x, y) \in \mathcal{R}$. We use the discussion of Case 3 to see $n_+(x, y) = 2$ and $[\Gamma(x, y)] = [\Gamma(\bar{x}, \bar{y})]$ for exactly one point $(\bar{x}, \bar{y}) \in \mathcal{D}$.

Case 2.1.5. Suppose that $y = 0$ and $x > 1$. We compute

$$\begin{aligned}(x_+, y_+) &= \left(\frac{1}{\sqrt{2x^2-1}}, -\frac{(1+2x^2)\sqrt{x^2-1}}{x\sqrt{2x^2-1}} \right), \\ (x_-, y_-) &= \left(-\frac{1}{\sqrt{2x^2-1}}, -\frac{(1+2x^2)\sqrt{x^2-1}}{x\sqrt{2x^2-1}} \right).\end{aligned}$$

Since $x_- = -x_+$ and $y_+ = y_-$, $n_+(x, y) = 2$. Since $x > 1$, $x_+ < 1$. Furthermore, one may verify that (x_+, y_+) is on the Jacobi locus L_+ . The map $x \rightarrow (x_+, y_+)$ parametrizes the Jacobi locus.

Suppose (x, y) is on the Jacobi locus. The discussion above shows $n_+(x, y) = 2$ since there exists a unique point $(\bar{x}, 0)$ with $\bar{x} > 1$ so $[\Gamma_+(x, y)] = [\Gamma_+(\bar{x}, 0)]$.

The complement in \mathcal{S} of the discriminant locus \mathcal{D} , the ray \mathcal{R} , and the Jacobi locus L_+ consists of 4 open disjoint regions pictured below in Fig. 2.3.

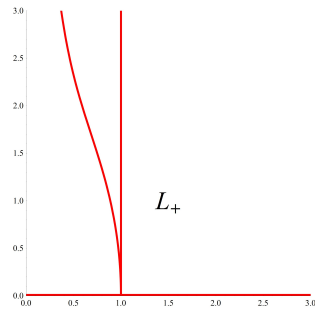


Fig. 2.2 ↑

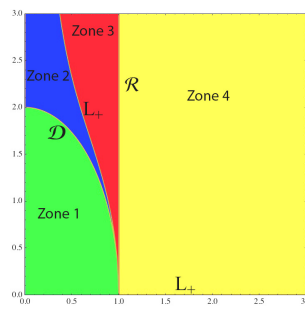


Fig. 2.3 ↑

Zone 1 is in green, Zone 2 is in blue, Zone 3 is in red, and Zone 4 is in yellow. The boundary between Zone 1 and Zone 2 is \mathcal{D} , the boundary between Zone 2 and Zone 3 is L_+ , and the boundary between Zone 3 and Zone 4 is \mathcal{R} . If $(x, y) \in \mathcal{S}$ and if $(\bar{x}, \bar{y}) \in \mathcal{S}$, then we say that $(x, y) \sim (\bar{x}, \bar{y})$ if and only if $[\Gamma_+(x, y)] = [\Gamma_+(\bar{x}, \bar{y})]$. Let \mathcal{O} be the union of Zone 2, Zone 3, and Zone 4. The analysis above shows that if $(x, y) \in \mathcal{O}$ and if $(x, y) \sim (\bar{x}, \bar{y})$, then $(\bar{x}, \bar{y}) \in \mathcal{O}$ as well.

We examine a specific example:

$$\begin{array}{ccc} (x, y) & (x_+(x, y), y_+(x, y)) & (x_-(x, y), y_-(x, y)) \\ (0.1, 2.2), & (0.636041, -10.0394), & (1.56535, -9.72988), \\ (0.636041, 10.0394), & (0.1, -2.2), & (1.56535, 9.72988), \\ (1.56535, 9.72988), & (0.1, -2.2), & (-0.636041, -10.0394). \end{array}$$

We adjust the signs and define $Q_{\pm}^2(x, y) := (x_{\pm}(x, y), -y_{\pm}(x, y))$ on Zone 2. The analysis performed above shows that Q_{\pm}^2 can not cross zone boundaries. Since $Q_+^2(0.1, 2.2)$ takes values in Zone 3 and $Q_-^2(0.1, 2.2)$ takes values in Zone 4, the same is true on all of Zone 2. Thus $n_+(x, y) = 3$ on Zone 2. A similar analysis pertains on Zone 3 and Zone 4.

Let Equation (2.d) define Y_+ . Let \mathcal{C} be the closure of Zone 1 and Zone 2:

$$\mathcal{C} := \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq Y_+(x)\}.$$

The following is now immediate from our discussion:

Lemma 2.5. *Let $[\Gamma] \in \mathfrak{Z}_+$. There is a unique $(x, y) \in \mathcal{C}$ so $[\Gamma_+(x, y)] = [\Gamma]$.*

The proof of Theorem 1.4 if $\rho > 0$. We now turn to the question of defining invariants which completely detect \mathfrak{Z}_+ . We change variables slightly to define

$$\begin{aligned} \Omega_+(x, y) &:= (\psi_3(\Gamma_+(x, y)), 4\psi_3(\Gamma_+(x, y)) - \Psi_3(\Gamma_+(x, y)) - 18) \\ &= (2 + x^{-2} + 4x^2 + y^2, -x^{-2}(x^2 - 1)^2(-4 + 4x^2 + y^2)). \end{aligned}$$

Let $(x, y) \equiv (\bar{x}, \bar{y})$ if and only if $\Omega_+(x, y) = \Omega_+(\bar{x}, \bar{y})$. To complete the proof of Theorem 1.4 for \mathfrak{Z}_+ , we must show that if $(x, y) \in \mathcal{C}$, $(\bar{x}, \bar{y}) \in \mathcal{C}$, and $(x, y) \equiv (\bar{x}, \bar{y})$ then $(x, y) = (\bar{x}, \bar{y})$. We observe that the curves $y \rightarrow \Omega_+(x, y)$ for fixed x are rays. We introduce three pictures

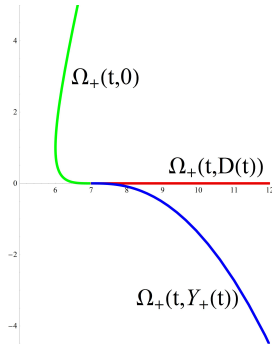


Fig. 2.4 \uparrow

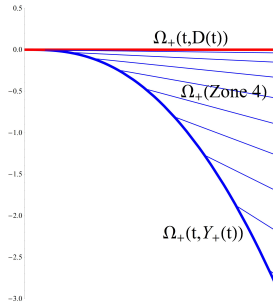


Fig. 2.5 \uparrow

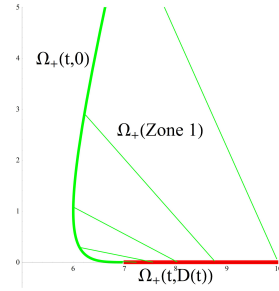


Fig. 2.6 \uparrow

Fig. 2.4 gives the boundary of the image of \mathcal{C} . The upper curve in the first quadrant is the image of the Jacobi locus, the image of the discriminant locus lies along the horizontal axis, and the lower curve in the fourth quadrant is the image of the x -axis for $0 < x < 1$.

$$\begin{aligned} \Omega_+(t, 0) &= (4t^2 + t^{-2} + 2, 4t^{-2}(1 - t^2)^3), \\ \Omega_+(t, D(t)) &= (t^{-2} + 6, 0), \\ \Omega_+(t, Y_+(t)) &= (6 + 2(t^2 + t^4)^{-1}, (t^4 + t^6)^{-1}(t^2 - 1)^3). \end{aligned}$$

It is immediate that $\Omega_+(t, 0)$ lies in the 1st quadrant, $\Omega_+(t, D(t))$ lies along the horizontal axis, and $\Omega_+(t, Y_+(t))$ lies in the 4th quadrant. If $(x, y) \in C$, we may represent $(x, y) = (t, s \cdot Y_+(t))$ for some $s \in [0, 1]$. The curve $s \rightarrow \Omega_+(t, s \cdot Y_+(t))$ for fixed t and $s \in [0, 1]$ is a straight line which runs from the upper curve in the first quadrant down to the corresponding point horizontal axis and then down to the lower curve. The singular point $(1, 0)$ corresponds to the point $(7, 0)$ on the horizontal axis where the upper curve intersects the lower curve. If y lies in the closure of Zone 1, then $\Omega_+(x, y)$ lies in the 4th quadrant; if y lies in the closure of Zone 2, then $\Omega_+(x, y)$ lies in the 1st quadrant. Suppose $(x, y) \equiv (\bar{x}, \bar{y})$ for $0 < x \leq 1$ and $0 \leq y \leq Y_+(x)$. Our analysis shows that if $x = 1$ then $(x, y) = (\bar{x}, \bar{y}) = (1, 0)$. Furthermore, (x, y) belongs to the closure of Zone 1 (resp. Zone 2) if and only if (\bar{x}, \bar{y}) belongs to the closure of Zone 1 (resp. Zone 2). We may therefore examine the two cases seriatim.

Instead of examining the image of Zone 2, we may equivalently examine the image of Zone 4. This is pictured in Fig. 2.5. The upper boundary is the x axis; this is the curve $\Omega_+(1, y)$ for $y \geq 0$. The lower boundary is the curve $\gamma(x) := \Omega_+(x, 0)$ where $x \geq 1$. The straight lines are $\Omega(x_n, y)$ for $y \geq 1$ and suitably chosen x_n . Our task is to show these do not intersect. We must show the first coordinate of γ increases monotonically, the second coordinate of γ decreases monotonically, the slope of γ decreases monotonically, and the slope of the lines leaving γ decrease monotonically and are always strictly greater than the slope of γ . Let $\gamma = (\varrho_1, \varrho_2)$. We have $\gamma(x) = (2 + x^{-2} + 4x^2, 4x^{-2}(1 - x^2)^3)$. As desired, the first coordinate increases monotonically and the second coordinate decreases monotonically. One computes that

$$\begin{aligned} \text{Slope of } \gamma &= \left. \frac{\partial_x \varrho_2(x, y)}{\partial_x \varrho_1(x, y)} \right|_{y=0} = -\frac{4(x^2 - 1)^2}{2x^2 - 1}, \\ \text{Slope of line from } \gamma &= \left. \frac{\partial_y \varrho_2(x, y)}{\partial_y \varrho_1(x, y)} \right|_{y=0} = -\frac{(x^2 - 1)^2}{x^2}. \end{aligned}$$

These are monotonically decreasing for $x \geq 1$, negative, and as desired the slope of γ is more negative than the slope of the line from γ . This completes our analysis of Zone 4.

We now turn our analysis to Zone 1 and refer to Fig. 2.6. The lower boundary is the x axis; this is the curve $\Omega_+(x, 2\sqrt{1 - x^2})$ and the upper boundary is the curve $\Omega_+(x, 0)$. The straight lines are the lines (x_n, y) for suitable values of x_n where $0 \leq y \leq 2\sqrt{1 - x_n^2}$. And if the picture is to be believed, the full rays do not intersect. Let $\tau(x) = 2\sqrt{1 - x^2}$ so that $(x, \tau(x))$ parametrizes the discriminant locus. We have $(\varrho_1, \varrho_2) = \Omega_+(x, \tau(x)) = (6 + x^{-2}, 0)$ and $\left. \frac{\partial_y \varrho_1(x, y)}{\partial_y \varrho_2(x, y)} \right|_{y=\tau(x)} = -\frac{(-1+x^2)^2}{x^2}$. Thus as x increases from 0 to 1, ϱ_1 monotonically decreases from ∞ to 7 and the slope of the straight lines monotonically increases from $-\infty$ to 0. Thus the picture really is as drawn and the rays do not intersect. This completes the proof of Theorem 1.4 in the positive definite setting. \square

2.2. The moduli space \mathfrak{Z}_- . We suppose that ρ is negative definite and recall our parametrization:

$$\Gamma_-(x, y) := \{\Gamma_{11}^1 = x - \frac{1}{x}, \Gamma_{11}^2 = 0, \Gamma_{12}^1 = 0, \Gamma_{12}^2 = x, \Gamma_{22}^1 = x, \Gamma_{22}^2 = y\}.$$

We recall some notation established previously and set

$$\begin{aligned} p_-(x, y) &:= \psi_3(\Gamma_-(x, y)) = 2 - 4x^2 - \frac{1}{x^2} - y^2, \\ P_-(x, y) &:= \Psi_3(\Gamma_-(x, y)) = 4x^4 + x^2(y^2 - 4) + \frac{y^2}{x^2} + 2(-y^2 + 1), \\ J_-(x, y) &:= \det(\Theta'_-) = \frac{4(x^2+1)y(4x^6+x^4y^2-x^2(y^2+3)+1)}{x^5}. \end{aligned}$$

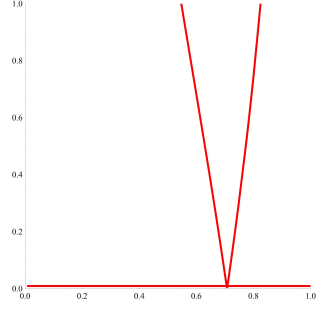


Fig. 2.7 ↑

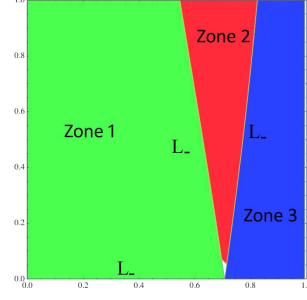


Fig. 2.8 ↑

The solutions of $J_-(x, y) = 0$ are the Jacobi-locus. This includes the ray $y = 0$ and the graph $(x, Y_-(x))$ pictured in Figure 2.7 with a corner at $(\frac{1}{\sqrt{2}}, 0)$ where

$$y = Y_-(x) = \sqrt{\frac{1 + 4x^6 - 3x^2}{x^2 - x^4}} \text{ for } 0 < x < 1.$$

The Jacobi locus divides \mathcal{S} into 3 zones. Zone 1 is in green; it lies below the Jacobi locus for $x < \sqrt{2}^{-1}$. Zone 2 is in red; it lies above the Jacobi locus. Zone 3 is the remainder. It lies below the Jacobi locus for $\sqrt{2}^{-1} \leq x < 1$. It also contains the half-plane $x \geq 1$.

We use Equation (2.f) to define $T_{u,v}$. Setting $\{T_{u,v}^* \Gamma\}_{12}^1 = 0$ yields the following cubic equation which is analogous to Equation (2.g) in the positive definite setting:

$$(1 + x^2)u^2 + uxy - x^2 = 0.$$

There is no discriminant locus and we obtain two solutions

$$u_{\pm} = x \frac{-y \pm \sqrt{4(1 + x^2) + y^2}}{2(1 + x^2)},$$

which we use to define x_{\pm} and y_{\pm} as before. We now let $n_-(x, y)$ be the number of distinct points (\bar{x}, \bar{y}) so $[\Gamma_-(x, y)] = [\Gamma_+(\bar{x}, \bar{y})]$. We analyze cases as follows.

Case 2.2.1. We examine the singular point $(\frac{1}{\sqrt{2}}, 0)$. This is where the Jacobi locus intersects the horizontal axis in a corner. We show $n_-(\frac{1}{\sqrt{2}}, 0) = 1$ by computing:

$$\begin{aligned} (x_+(\frac{1}{\sqrt{2}}, 0), y_+(\frac{1}{\sqrt{2}}, 0)) &= (-\frac{1}{\sqrt{2}}, 0), \\ (x_-(\frac{1}{\sqrt{2}}, 0), y_-(\frac{1}{\sqrt{2}}, 0)) &= (+\frac{1}{\sqrt{2}}, 0). \end{aligned}$$

Case 2.2.2. Suppose that $(x, y) = (x, 0)$ lies on the horizontal axis but $x \neq \frac{1}{\sqrt{2}}$. We show $(x_{\pm}(x, 0), y_{\pm}(x, 0))$ lies on the Jacobi locus as follows. If $x = \sqrt{t}$, then:

$$\begin{aligned} x_{\pm}^2(\sqrt{t}, 0) &= \frac{1}{1 + 2t}, \quad y_{\pm}^2(\sqrt{t}, 0) = \frac{1 - 3t + 4t^3}{t + 2t^2}, \\ 4x_{\pm}^6 + x_{\pm}^4 y_{\pm}^2 - x_{\pm}^2(y_{\pm}^2 + 3) + 1 &= 0. \end{aligned}$$

Consequently, $n(x, y) = 2$ on the horizontal axis minus $(\frac{1}{\sqrt{2}}, 0)$ and $n(x, y) = 2$ on the Jacobi locus minus $(\frac{1}{\sqrt{2}}, 0)$; every point on the horizontal axis for $x < \frac{1}{\sqrt{2}}$ corresponds to a point of the Jacobi locus (\bar{x}, \bar{y}) for $\bar{x} > \frac{1}{\sqrt{2}}$ while every point on the horizontal axis for $x > \frac{1}{\sqrt{2}}$ corresponds to a point on the Jacobi locus (\bar{x}, \bar{y}) for $\bar{x} < \frac{1}{\sqrt{2}}$; the two regions are reversed.

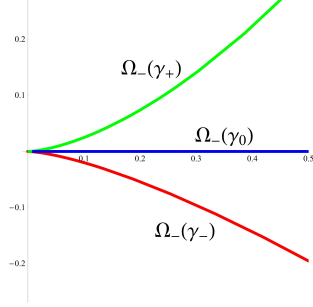
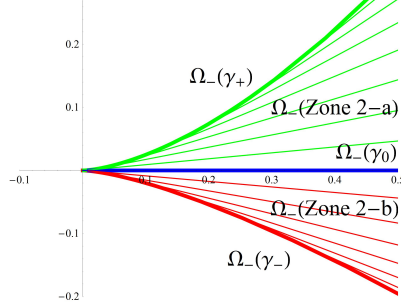
Case 2.2.3. The discussion above shows that no point in the interior of Zone 1, Zone 2, or Zone 3 is equivalent to a point on the Jacobi locus or on the horizontal axis in the moduli space. We compute:

$$\begin{array}{ccc} (x, y) & (x_+(x, y), y_+(x, y)) & (x_-(x, y), y_-(x, y)) \\ (1/\sqrt{2}, 100), & (-0.00999825, 0.706983), & (1.4139, -99.9775), \\ (0.00999825, 0.706983), & (-0.707107, 100.), & (1.4139, 99.9775), \\ (1.4139, 99.9775), & (-0.00999825, -0.706983), & (0.707107, -100.). \end{array}$$

We now conclude every point in Zone 1 is equivalent to a point in Zone 2 and to a point in Zone 3; so $n(x, y) = 3$ for these points.

We now examine the image of Zone 2. It is convenient to decompose Zone 2 into two parts where Zone 2-a lies above the Jacobi locus for $0 < x \leq \sqrt{2}^{-1}$ and Zone 2-b lies above the Jacobi locus for $\sqrt{2}^{-1} \leq x < 1$. Again, we renormalize setting

$$\begin{aligned} \Omega_-(x, y) &:= (-2 - \psi_3(\Gamma_-(x, y)), 2\Psi_3(\Gamma_-(x, y)) + \psi_3(\Gamma_+(x, y))) \\ &= (4x^2 + x^{-2} + y^2 - 4, x^{-2}(2x^2 - 1)(4x^4 + x^2(y^2 - 4) - 2y^2 + 1)). \end{aligned}$$

Fig. 2.9 \uparrow Fig. 2.10 \uparrow

The x -axis is also part of the Jacobi locus in this instance. Figure 2.9 shows the image of the graph of Y_- of the Jacobi locus and the line $x = \sqrt{2}^{-1}$. The upper curve γ_+ is the image of the graph of Y_- of the Jacobi locus for $0 < x < \sqrt{2}^{-1}$, the curve lying along the horizontal axis γ_0 is the image of the ray $(\sqrt{2}^{-1}, t)$, and the lower curve γ_- is the image of the graph of Y_- of the Jacobi locus for $\sqrt{2}^{-1} < x < 1$. More explicitly,

$$\begin{aligned} \gamma_+(t) &= 2\{t^2(1-t^2)\}^{-1}((1-2t^2)^2, \frac{(1-2t^2)^3}{t^2}) \text{ for } 0 < t \leq \sqrt{2}^{-1}, \\ \gamma_0(t) &= (t^2, 0), \text{ for } 0 \leq t, \\ \gamma_-(t) &= 2\{t^2(1-t^2)\}^{-1}((1-2t^2)^2, \frac{(1-2t^2)^3}{t^2}) \text{ for } \sqrt{2}^{-1} \leq t < 1. \end{aligned}$$

Thus the picture actually is as drawn. Fig. 2.10 includes the lines $\Omega(x_n, Y_+(x_n) + s)$ for suitably chosen values of x_n and $s \in [0, \infty)$. This will complete the proof if the picture can be believed as this forms a pencil of rays that do not intersect. Since we are on the Jacobi locus, the rank of the Jacobian of Ω_- is 1. Let $F_-(t, s) := \Omega_-(t, Y_-(t) + s) = (\varrho_1(t, s), \varrho_2(t, s))$. We compute

$$\frac{\partial_t \varrho_2}{\partial_t \varrho_1} \Big|_{s=0} = \frac{\partial_s \varrho_2}{\partial_s \varrho_1} \Big|_{s=0} = -5 + 2t^{-2} + 2t^2.$$

As t ranges from 0 to $\sqrt{2}^{-1}$ to 1, $F_-(t, 0)$ ranges from (∞, ∞) to $(0, 0)$ to $(\infty, -\infty)$ and traverses the curve $\gamma = \gamma_+ \cup \gamma_-$ counterclockwise. The slope ranges from $+\infty$ to 0 to $-\infty$ and decreases monotonically. The lines $s \rightarrow F_-(t_0, s)$ are tangent to the curve and their slopes also decrease monotonically. Thus the situation is as

depicted in Fig. 2.10 and these curves form a pencil of rays that do not intersect. This completes the proof of Theorem 1.4 in the negative definite setting.

3. THE MODULI SPACE \mathfrak{Z}_0

Let $\Gamma \in \mathcal{Z}_0 \subset \mathbb{R}^6$ be a Christoffel symbol so that the associated Ricci tensor ρ has signature $(1, 1)$. After making a linear change of coordinates, we may assume

$$\rho = dx^1 \otimes dx^2 + dx^2 \otimes dx^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This normalizes the change of coordinates to the appropriate orthogonal group

$$\begin{aligned} SO(1, 1) &:= \{T_a \in GL(2, \mathbb{R}) : T_a : (x^1, x^2) \rightarrow (ax^1, a^{-1}x^2) \text{ for } a \neq 0\}, \\ O(1, 1) &:= SO(1, 1) \cup \tilde{T} \cdot SO(1, 1) \text{ for } \tilde{T} : (x^1, x^2) \rightarrow (x^2, x^1). \end{aligned} \quad (3.a)$$

In this case we do not have a single parametrization of \mathfrak{Z}_0 like we do for \mathfrak{Z}_\pm ; there are in fact two parametrizations which give rise to the two boundary curves for \mathfrak{Z}_0 that must be glued along their common border $xy = 1$.

We set $\Gamma_{11}^1 = \Gamma_{12}^2 + \alpha$ and $\Gamma_{22}^2 = \Gamma_{12}^1 + \beta$ to obtain

$$\begin{aligned} \rho &= \begin{pmatrix} \beta\Gamma_{11}^2 + \alpha\Gamma_{12}^2 & \Gamma_{12}^1\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^1 \\ \Gamma_{12}^1\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^1 & \beta\Gamma_{12}^1 + \alpha\Gamma_{22}^1 \end{pmatrix} \\ &= \begin{pmatrix} (\Gamma_{12}^2, \Gamma_{11}^2) \cdot (\alpha, \beta) & (\Gamma_{12}^2, \Gamma_{11}^2) \cdot (\Gamma_{12}^1, -\Gamma_{22}^1) \\ (\Gamma_{12}^2, \Gamma_{11}^2) \cdot (\Gamma_{12}^1, -\Gamma_{22}^1) & (\Gamma_{22}^1, \Gamma_{12}^1) \cdot (\alpha, \beta) \end{pmatrix}. \end{aligned}$$

Suppose $(\alpha, \beta) \neq (0, 0)$. Then $(-\beta, \alpha)$ is a non-zero vector perpendicular to (α, β) . Setting $\rho_{11} = \rho_{22} = 0$ yields $(\Gamma_{12}^2, \Gamma_{11}^2) = a(-\beta, \alpha)$ and $(\Gamma_{22}^1, \Gamma_{12}^1) = b(-\beta, \alpha)$ for some $a, b \in \mathbb{R}$. This implies that

$$\rho_{12} = (\Gamma_{12}^2, \Gamma_{11}^2) \cdot (\Gamma_{12}^1, -\Gamma_{22}^1) = ab(-\beta, \alpha) \cdot (\alpha, \beta) = 0$$

and $\rho = 0$. Thus $(\alpha, \beta) = (0, 0)$. This yields the ansatz:

$$\Gamma_{11}^1 = \Gamma_{12}^2 = x, \quad \Gamma_{12}^1 = \Gamma_{22}^2 = y, \quad \Gamma_{11}^2\Gamma_{22}^1 = xy - 1 \quad \text{for some } (x, y) \in \mathbb{R}^2.$$

The map T_a of Equation (3.a) multiplies Γ_{11}^2 by a^3 and Γ_{22}^1 by a^{-3} . Suppose first $xy - 1 > 0$. This implies $\Gamma_{11}^2\Gamma_{22}^1 > 0$. By choosing a appropriately, we may assume $\Gamma_{11}^2 = \Gamma_{22}^1$ is positive and obtain the parametrization:

$$\begin{aligned} \Gamma_{0,1}(x, y) &:= \{\Gamma_{11}^1 = \Gamma_{12}^2 = x, \quad \Gamma_{11}^2 = \sqrt{xy - 1}, \\ &\quad \Gamma_{22}^1 = \sqrt{xy - 1}, \quad \Gamma_{12}^1 = \Gamma_{22}^2 = y\}. \end{aligned} \quad (3.b)$$

On the other hand, if $xy - 1 < 0$, then $\Gamma_{11}^2\Gamma_{22}^1$ is negative and again by choosing a appropriately to define T_a , we may assume $\Gamma_{11}^2 = -\Gamma_{22}^1$ and Γ_{11}^2 is positive and obtain the parametrization:

$$\begin{aligned} \Gamma_{0,2}(x, y) &:= \{\Gamma_{11}^1 = \Gamma_{12}^2 = x, \quad \Gamma_{11}^2 = \sqrt{1 - xy}, \\ &\quad \Gamma_{22}^1 = -\sqrt{1 - xy}, \quad \Gamma_{12}^1 = \Gamma_{22}^2 = y\}. \end{aligned} \quad (3.c)$$

The case $xy = 1$ is exceptional. Set

$$\Theta_{0,i}(x, y) := (\psi_3(\Gamma_{0,i}(x, y)), \Psi_3(\Gamma_{0,i}(x, y))) \text{ for } i = 1, 2.$$

We compute:

$$\begin{aligned} \Theta_{0,1}(x, y) &= (-2 + 8xy, 1 - 4xy + 8x^2y^2 - 4(x^3 + y^3)\sqrt{xy - 1}), \\ \Theta_{0,2}(x, y) &= (-2 + 8xy, 1 - 4xy + 8x^2y^2 + 4(x^3 - y^3)\sqrt{1 - xy}). \end{aligned} \quad (3.d)$$

We distinguish cases:

Case 3.1. Suppose $xy > 1$. Adopt the notation of Equation (3.b). By interchanging x^1 and x^2 we can interchange the roles of x and y to assume $|x| \geq |y|$. It is natural to make the change of variables:

$$(s(x, y), t(x, y)) := (xy, x/\sqrt{s}) \text{ and } (x(s, t), y(s, t)) := (\sqrt{st}, \sqrt{st}^{-1}). \quad (3.e)$$

To ensure $xy > 1$, we take $s > 1$. To ensure $|x| \geq |y|$, we take $|t| \geq 1$. Then:

$$\Theta_{0,1}(s(x, y), t(x, y)) = (-2 + 8s, 1 - 4s + 8s^2 - 4s^{3/2}\sqrt{s-1}(t^3 + t^{-3})).$$

We may recover s from the first coordinate and $t^3 + t^{-3}$ from the second coordinate. The function $t \rightarrow t^3 + t^{-3}$ is monotonically increasing and negative for $t \in (-\infty, -1)$ and monotonically increasing and positive for $t \in (1, \infty)$. Consequently we may recover t as well and conclude Θ_0 is 1-1 on this region. We give below in Fig.3.1 a picture of this region. The vertical straight lines correspond to appropriate values of $s = s_n$ taken to be constant. The upper rays are given by $t \leq -1$ and the lower rays by $t \geq 1$.

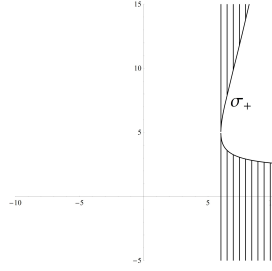


Fig. 3.1↑

Case 3.2. Suppose that $xy = 1$. We have $\Gamma_{11}^2 \Gamma_{22}^1 = 0$. If $\Gamma_{22}^1 = 0$, we obtain the point $(6, 5)$ which was missing from the boundary curve discussed in Case 3.1 above. On the other hand, if Γ_{22}^1 is non-zero, we can normalize $\Gamma_{22}^1 = 1$ and obtain $\Psi_3 = 5 - 4xy^{-2} = 5 - 4x^3$ where $x \neq 0$. This yields the remainder of the line $\psi_3 = 6$. This is pictured below in Fig. 3.2.

Case 3.3. Suppose $0 < xy < 1$. The map $(x^1, x^2) \rightarrow (-x^2, -x^1)$ replaces (x, y) by $(-y, -x)$. Thus as x and y have the same sign, we may assume without loss of generality $x > 0$ and $y > 0$ in using the parametrization $\Gamma_{0,2}$ of Equation (3.c). We make the change of variables $x = \sqrt{st}$, $y = \sqrt{s}/t$ given in Equation (3.e) where $0 < s < 1$. We take $t > 0$ since $x > 0$. We use Equation (3.d) to obtain

$$\Theta_{0,2}(s(x, y), t(x, y)) = (-2 + 8s, 1 - 4s + 8s^2 + 4s^{\frac{3}{2}}\sqrt{1-s}(t^3 - t^{-3})).$$

The parameter s is determined by ψ_3 and $t^3 - t^{-3}$ is determined by Ψ_3 . Since $t^3 - t^{-3}$ is positive and monotonic increasing for $t \in (0, \infty)$, we may compute t . Consequently (ψ_3, Ψ_3) is 1-1 in this range and completely fills up the region $-2 < \psi_3 < 6$ and $-\infty < \Psi_3 < \infty$. This is pictured below in Fig. 3.3. Note that if $0 < xy < 1$ then $-2 < xy < 6$.

Case 3.4. Suppose $xy = 0$. We set $x = 0$ in $\Gamma_{0,2}(x, y)$. We obtain $\psi_3 = -2$ and $\Psi_3 = 1 - 4y^3$. So this fills up the complete line. This is pictured below in Fig. 3.4:

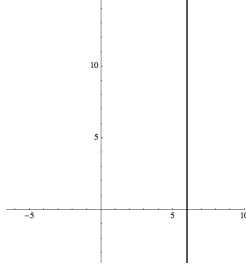


Fig. 3.2↑

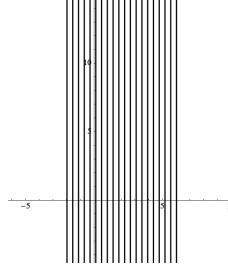


Fig. 3.3↑

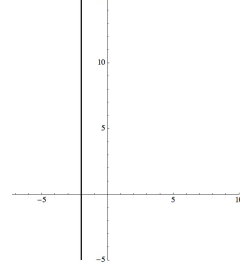


Fig. 3.4↑

Case 3.5. Suppose $xy < 0$. The map $(x^1, x^2) \rightarrow (-x^2, -x^1)$ preserves the normalization. So we can assume $|x| \geq |y|$. We now make the change of variables

$$(s(x, y), t(x, y)) := (-xy, x/\sqrt{s}) \text{ and } (x(s, t), y(s, t)) := (\sqrt{st}, -\sqrt{st}^{-1}).$$

We then have $s > 0$, and $|t| \geq 1$. We compute

$$\Theta_{0,2}(\Gamma_{0,2}(x(t, s), y(t, s))) = (-2 - 8s, 1 + 4s + 8s^2 + 4s^{3/2}(t^3 + t^{-3})\sqrt{1+s}).$$

Thus s is determined and $t^3 + t^{-3}$ is determined. This is negative and monotonically increasing for $-\infty < t \leq -1$ and positive and monotonically increasing for $1 \leq t < \infty$ and hence t is determined. Thus Θ is 1-1. This region is pictured below in Fig. 3.5; the full region $\Theta_0(\mathfrak{Z}_0)$ is pictured in Fig. 3.6:

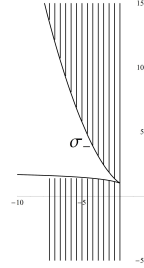


Fig. 3.5↑

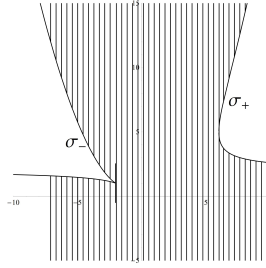


Fig. 3.6↑

4. THE PROOF OF THEOREM 1.5 AND THEOREM 1.6

4.1. The boundary curves. In this section, we will prove Theorem 1.5. The boundary of \mathfrak{Z}_0 consists of two pieces. We shall identify the right hand component of the boundary of \mathfrak{Z}_0 with the left hand boundary of \mathfrak{Z}_+ and the left hand component of the boundary of \mathfrak{Z}_0 with the right hand boundary of \mathfrak{Z}_- .

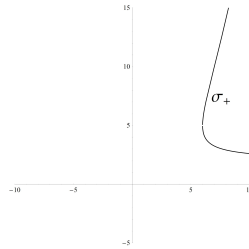


Fig. 4.1↑

The right boundary of \mathfrak{Z}_0 and the left boundary of \mathfrak{Z}_+ . We recall the parametrizations defined previously

$$\begin{aligned}\Gamma_+(x, y) &:= \left\{ \begin{array}{lll} \Gamma_{11}^1 = x + \frac{1}{x}, & \Gamma_{11}^2 = 0, & \Gamma_{12}^1 = 0, \\ \Gamma_{12}^2 = x, & \Gamma_{22}^1 = x, & \Gamma_{22}^2 = y. \end{array} \right\} \\ \Gamma_{0,1}(x, y) &:= \left\{ \begin{array}{lll} \Gamma_{11}^1 = x, & \Gamma_{11}^2 = \sqrt{xy-1}, & \Gamma_{12}^1 = y, \\ \Gamma_{12}^2 = x, & \Gamma_{22}^1 = \sqrt{xy-1}, & \Gamma_{22}^2 = y. \end{array} \right\}\end{aligned}$$

The left boundary of $\Theta_+(\mathfrak{Z}_+)$ is the curve $\sigma_+(u) := \Theta_+(\Gamma_+(u, 0))$ and the right boundary of $\Theta_{0,1}(\mathfrak{Z}_0)$ is the curve $\sigma_{0,1}(v) := \Theta_{0,1}(\Gamma_{0,1}(v, v))$. We have (see Equation (1.c)) that:

$$\begin{aligned}\sigma_+(u) &= (2 + u^{-2} + 4u^2, 2 + 4u^2 + 4u^4) \text{ for } 0 < u, \\ \sigma_{0,1}(v) &= (-2 + 8v^2, 1 - 4v^2 + 8v^4 - 8v^3\sqrt{v^2-1}) \text{ for } |v| > 1.\end{aligned}$$

We set $-2 + 8v^2 = 2 + u^{-2} + 4u^2$ to see $8v^2 = 4 + u^{-2} + 4u^2 = (2u + u^{-1})^2$. Thus

$$\sqrt{v^2-1} = 2^{-3/2}\sqrt{8v^2-8} = 2^{-3/2}\sqrt{(2u + u^{-1})^2 - 8} = 2^{-3/2}\sqrt{(2u - u^{-1})^2}.$$

There are two possible choices of the square root.

- (1) We set $v = 2^{-3/2}(2u + u^{-1}) \geq 1$, $\sqrt{v^2-1} = 2^{-3/2}(u^{-1} - 2u)$, $0 < u < \frac{1}{\sqrt{2}}$ to obtain $1 - 4v^2 + 8v^4 - 8v^3\sqrt{v^2-1} = 2 + 4u^2 + 4u^4$. This is the bottom curve marked in red in Fig. 4.1.
- (2) We set $v = -2^{-3/2}(2u + u^{-1}) \leq -1$, $\sqrt{v^2-1} = 2^{-3/2}(2u - u^{-1})$, $\frac{1}{\sqrt{2}} < u$ to obtain $1 - 4v^2 + 8v^4 - 8v^3\sqrt{v^2-1} = 2 + 4u^2 + 4u^4$. This is the upper curve marked in blue in Fig. 4.1 above.

The left boundary of \mathfrak{Z}_0 and the right boundary of \mathfrak{Z}_- . We recall the parametrization defined previously

$$\begin{aligned}\Gamma_-(x, y) &:= \left\{ \begin{array}{lll} \Gamma_{11}^1 = x - \frac{1}{x}, & \Gamma_{11}^2 = 0, & \Gamma_{12}^1 = 0, \\ \Gamma_{12}^2 = x, & \Gamma_{22}^1 = x, & \Gamma_{22}^2 = y. \end{array} \right\} \\ \Gamma_{0,2}(x, y) &:= \left\{ \begin{array}{lll} \Gamma_{11}^1 = x, & \Gamma_{11}^2 = \sqrt{1-xy}, & \Gamma_{12}^1 = y, \\ \Gamma_{12}^2 = x, & \Gamma_{22}^1 = -\sqrt{1-xy}, & \Gamma_{22}^2 = y. \end{array} \right\}\end{aligned}$$

The right boundary of $\Theta_-(\mathfrak{Z}_-)$ is the curve $\sigma_-(u) := \Theta_-(\Gamma_-(u, 0))$ and the left boundary of $\Theta_{0,2}(\mathfrak{Z}_0)$ is the curve $\sigma_{0,2}(v) := \Theta_{0,2}(\Gamma_{0,2}(v, -v))$. We have

$$\begin{aligned}\sigma_-(u) &= (2 - 4u^2 - u^{-2}, 4u^4 - 4u^2 + 2), \text{ for } 0 < u \\ \sigma_{0,2}(v) &= (-2 - 8v^2, 1 + 4v^2 + 8v^4 + 8v^3\sqrt{1+v^2}), \text{ for } v \in \mathbb{R}.\end{aligned}$$

We solve the first equation to see

$$8v^2 = (2u - u^{-1})^2 \text{ and } \sqrt{v^2+1} = 2^{-3/2}\sqrt{(2u - u^{-1})^2}.$$

Again, there are two choices of the square root.

- (1) We set $v = 2^{-3/2}(2u - u^{-1}) < 0$, $0 < u < \frac{1}{\sqrt{2}}$, to obtain the lower curve in Fig. 4.2 below: $1 + 4v^2 + 8v^4 + 8v^3\sqrt{1+v^2} = 2 - 4u^2 + 4u^4$.
- (2) We set $v = 2^{-3/2}(2u - u^{-1}) > 0$, $\frac{1}{\sqrt{2}} < u$ to obtain the upper curve in Fig. 4.2 below: $1 + 4v^2 + 8v^4 + 8v^3\sqrt{1+v^2} = 2 - 4u^2 + 4u^4$.

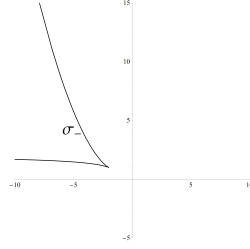


Fig. 4.2↑

4.2. The oriented moduli space. In this section, we shall prove Theorem 1.6. Let $\Gamma \in \mathbb{R}^6$ define a Christoffel symbol with non-degenerate Ricci tensor ρ . Assume $T^*\Gamma = \Gamma$ for $\text{id} \neq T \in \text{GL}(2, \mathbb{R})$.

Suppose ρ is positive definite and $\det(T) > 0$. We choose local coordinates so $\rho = \text{diag}(1, 1)$; T is then a rotation through an angle θ . We can diagonalize T over \mathbb{C} . Since $\det(T) = 1$, $T = \text{diag}(a, a^{-1})$ for $a \in S^1$. We have

$$(T^*\Gamma)_{ij}{}^k = a^\epsilon \Gamma_{ij}{}^k \text{ for } \epsilon = \pm 1 \pm 1 \pm 1 \in \{-3, -1, 1, 3\}.$$

If $\Gamma_{11}{}^2$ or $\Gamma_{22}{}^1$ is non-zero, we conclude $a^{\pm 3} = 1$. If one of the other Christoffel symbols is non-zero, we conclude $a^{\pm 1} = 1$. Thus either T is the identity or T is a rotation thru an angle of $\pm \frac{2\pi}{3}$. If $|\mathcal{G}^+(\Gamma)| \neq 1$, $T_1 := T_{\frac{2\pi}{3}}$ and $T_2 := T_{\frac{4\pi}{3}}$ must preserve Γ . We use the parametrization

$$\Gamma_+(x, y) := \{\Gamma_{11}{}^1 = x \pm \frac{1}{x}, \Gamma_{11}{}^2 = 0, \Gamma_{12}{}^1 = 0, \Gamma_{12}{}^2 = x, \Gamma_{22}{}^1 = x, \Gamma_{22}{}^2 = y\}$$

given in Definition 2.2 and compute

$$\begin{aligned} 0 &= (T_{\frac{2\pi}{3}}^* \Gamma)_{11}{}^2 = (8x)^{-1} \{-\sqrt{3} - 2\sqrt{3}x^2 - 3xy\}, \\ 0 &= (T_{\frac{4\pi}{3}}^* \Gamma)_{11}{}^2 = (8x)^{-1} \{\sqrt{3} + 2\sqrt{3}x^2 - 3xy\}. \end{aligned}$$

This implies $y = 0$ and $\sqrt{3} + 2\sqrt{3}x^2 = 0$. This is not possible. Thus $|\mathcal{G}^+(\Gamma)| = 1$ and $\text{GL}^+(2, \mathbb{R})$ acts without fixed points on \mathcal{Z}_+ .

Suppose ρ is positive definite and $\det(T) < 0$. We can choose coordinates so $\rho = \text{diag}(1, 1)$. Since $T^2 \in \mathcal{G}^+(\Gamma)$, we conclude $T^2 = \text{id}$ so T is a reflection. We can normalize the choice of coordinates so $T(x^1, x^2) = (x^1, -x^2)$ and, consequently, $(T^*\Gamma)_{12}{}^1 = -\Gamma_{12}{}^1$. Thus $\Gamma_{12}{}^1 = 0$. The discussion of Section 2 implies $\Gamma = \Gamma_+(x, y)$ for some (x, y) . A similar argument shows $\Gamma_{22}{}^2 = 0$ as well. Thus $y = 0$ and Γ belongs to the boundary of \mathfrak{C}_+ . Conversely, if Γ belongs to the boundary of \mathfrak{C}_+ , then $y = 0$ and it follows $T^*\Gamma_+(x, 0) = \Gamma_+(x, 0)$. This completes the proof of Theorem 1.6 if ρ is positive definite.

Suppose ρ is negative definite and $\det(T) > 0$. The same argument used in the positive definite setting shows that T is the identity or is a rotation thru an angle of $\pm \frac{2\pi}{3}$. This time, we use the parametrization

$$\Gamma_-(x, y) = \{\Gamma_{11}{}^1 = x - \frac{1}{x}, \Gamma_{11}{}^2 = 0, \Gamma_{12}{}^1 = 0, \Gamma_{12}{}^2 = x, \Gamma_{22}{}^1 = x, \Gamma_{22}{}^2 = y\}$$

and compute

$$\begin{aligned} 0 &= (T_{\frac{2\pi}{3}}^* \Gamma)_{11}{}^2 = (8x)^{-1} \{-2\sqrt{3}x^2 - 3xy + \sqrt{3}\} \\ 0 &= (T_{\frac{4\pi}{3}}^* \Gamma)_{11}{}^2 = (8x)^{-1} \{2\sqrt{3}x^2 - 3xy - \sqrt{3}\}. \end{aligned}$$

This implies $y = 0$ and $x = \frac{1}{\sqrt{2}}$. This is, of course, the cusp point of the boundary

$$\Gamma = \frac{1}{\sqrt{2}} \{\Gamma_{11}{}^1 = -1, \Gamma_{11}{}^2 = 0, \Gamma_{12}{}^1 = 0, \Gamma_{12}{}^2 = 1, \Gamma_{22}{}^1 = 1, \Gamma_{22}{}^2 = 0\}.$$

And we verify explicitly $T_\theta^* \Gamma_{\text{csp}} = \Gamma_{\text{csp}}$ for $\theta = \frac{2\pi}{3}$ or $\theta = \frac{4\pi}{3}$. Thus $|\mathcal{G}^+(\Gamma)| = 3$ if $[\Gamma] = [\Gamma_{\text{csp}}]$ and $|\mathcal{G}^+(\Gamma)| = 1$ otherwise.

Suppose ρ is negative definite and $\det(T) < 0$. Again, we can perform a rotation to assume $T(x^1, x^2) = (x^1, -x^2)$. The same argument as that given in the positive definite setting shows $T^* \Gamma_-(x, y) = \Gamma_-(x, y)$ if and only if $y = 0$, i.e. Γ belongs to the boundary of \mathfrak{C}_- . Thus $\mathcal{G}(\Gamma) - \mathcal{G}^+(\Gamma)$ is non-zero if and only if $y = 0$; $|\mathcal{G}(\Gamma)| = 2|\mathcal{G}^+(\Gamma)|$ in this instance. This completes the proof of Theorem 1.6 if ρ is negative definite.

Suppose ρ is indefinite and $\det(T) > 0$. We change coordinates to ensure $\rho = dx^1 \otimes dx^2 + dx^2 \otimes dx^1$. In this case, $T(x^1, x^2) = (ax^1, a^{-1}x^2)$ and we do not need to complexify. The equation $a^{\pm 1} = 1$ or $a^{\pm 3} = 1$ implies $a = 1$ and T is the identity.

Suppose ρ is indefinite and $\det(T) < 0$. We have $T(x^1, x^2) = (ax^2, a^{-1}x^1)$ for $a \in \mathbb{R}$. We use the ansatz

$$\Gamma_{11}^1 = \Gamma_{12}^2 = x, \quad \Gamma_{12}^1 = \Gamma_{22}^2 = y, \quad \Gamma_{11}^2 \Gamma_{22}^1 = xy - 1,$$

for $(x, y) \in \mathbb{R}^2$ described in Section 3. We consider cases:

Case 4.1. Suppose $xy > 1$. We continue the discussion of Case 3.1 above and normalize $\Gamma_{11}^2 = \Gamma_{22}^1 = \sqrt{xy - 1}$. Then $T^* \Gamma_{11}^2 = a^3 \Gamma_{22}^1$ so $a^3 = 1$ and $a = 1$. We then have $T^* \Gamma_{11}^1 = \Gamma_{22}^2$ so $x = y$. Thus Γ lies on the curve $\sigma_{0,1}(t, t)$ which is in the right hand boundary of $\Theta_0(\mathfrak{Z}_0)$. Conversely, if $x = y$, then $T^* \Gamma = \Gamma$.

Case 4.2. Suppose $xy = 1$. We continue the discussion of Case 3.2 above. We have that $\Gamma_{11}^2 \Gamma_{22}^1 = 0$. Since T^* interchanges Γ_{11}^2 and Γ_{22}^1 up to a multiple, we have $\Gamma_{11}^2 = \Gamma_{22}^1 = 0$. We can normalize $x = y = 1$ to obtain the point $(6, 5)$ which was missing from the boundary curve discussed above; conversely this point is clearly invariant under the action of T .

Case 4.3. Suppose $0 < xy < 1$. We continue the discussion of Case 3.3 above. Let $\Gamma_{11}^2 = \sqrt{1 - xy}$ and $\Gamma_{22}^1 = -\sqrt{1 - xy}$. Since $T^* \Gamma_{11}^2 = a^3 \Gamma_{22}^1 = -a^3 \Gamma_{11}^2$, we have $a = -1$. But then $T^* \Gamma_{11}^1 = -\Gamma_{22}^2$ implies $x = -y$. This is not possible with $0 < xy < 1$.

Case 4.4. Suppose $0 = xy$. We continue the discussion of Case 3.4 above. Since T interchanges Γ_{11}^1 and Γ_{22}^2 up to a multiple, we have $x = y = 0$. We can normalize $\Gamma_{11}^2 = 1$ and $\Gamma_{22}^1 = -1$. This yields the cusp point $(-2, 1)$ of \mathfrak{C}_0 in \mathbb{R}^2 .

Case 4.5. Suppose $xy < 0$. We continue the discussion of Case 3.5 above. We can normalize $\Gamma_{11}^2 = \sqrt{1 - xy}$ and $\Gamma_{22}^1 = -\sqrt{1 - xy}$. This shows $a = -1$ so $x = -y$. This yields the boundary curve $\sigma_{0,2}$. This completes the proof of Assertions (1), (2) and (3) in Theorem 1.6.

4.3. Invariants detecting \mathfrak{Z}_ϵ^+ . Let $\chi(\Gamma)$ be the invariant of Equation (1.d):

$$\chi(\Gamma) := \rho(\Gamma_{ab}^b \Gamma_{ij}^k \rho_{kl}^3 \rho^{ij} dx^a \wedge dx^l, \text{dvol}).$$

This is an invariant of \mathfrak{Z}_ϵ^+ . We use the parametrizations $\Gamma_\pm(x, y)$ of Definition 2.2 and the parametrization $\Gamma_{0,1}$ and $\Gamma_{0,2}$ of Equations (3.b) and (3.c). We compute:

$$\begin{aligned} \chi_+(\Gamma_+(x, y)) &= yx^{-3} (4x^6 + x^4 y^2 + x^2 (y^2 - 3) - 1), \\ \chi_-(\Gamma_-(x, y)) &= yx^{-3} (-4x^6 - x^4 y^2 + x^2 (y^2 + 3) - 1), \\ \chi_0(\Gamma_{0,1}(x, y)) &= 8\sqrt{xy - 1} (y^3 - x^3), \\ \chi_0(\Gamma_{0,2}(x, y)) &= 8\sqrt{1 - xy} (x^3 + y^3). \end{aligned} \tag{4.a}$$

We recall Equation (2.c) that the Jacobian determinant is given by

$$\begin{aligned} J_-(x, y) &:= \frac{4(x^2+1)y(4x^6+x^4y^2-x^2(y^2+3)+1)}{x^5}, \\ J_+(x, y) &:= -\frac{4(x^2-1)y(4x^6+x^4y^2+x^2(y^2-3)-1)}{x^5}. \end{aligned} \quad (4.b)$$

Suppose ρ is positive definite. Excluding the line $x = 1$, the Jacobian determinant J_+ given in Equation (4.b) vanishes precisely on the x -axis where $y = 0$ or on the Jacobi Locus L_+ and these two curves are identified by Θ_+ . Thus we use Equation (4.a) to see that χ_+ vanishes precisely on those elements of \mathfrak{Z}_+ which are invariant under reversing the orientation. In particular, note that $\chi_+(x, y) := \chi(\Gamma_+(x, y))$ changes sign if we replace y by $-y$ (or x by $-x$) which reverses the orientation as it corresponds to the coordinate change $x^2 \rightarrow -x^2$ (or $x^1 \rightarrow -x^1$). This shows that (ψ_3, Ψ_3, χ) completely detect \mathfrak{Z}_+ . A picture of this surface in \mathbb{R}^3 is given below. It crosses the plane where the third coordinate (listed horizontally in stripes) vanishes; this is exactly the boundary of \mathfrak{Z}_+ . We have doubled this region with boundary to produce what looks like a smooth surface in \mathbb{R}^3 .

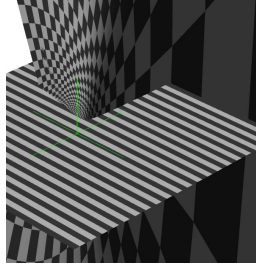


Fig. 4.3↑

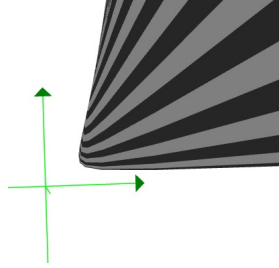


Fig. 4.4↑

Suppose ρ is negative definite. The analysis is essentially the same as in the positive definite setting. Let $\chi_-(x, y) := \chi(\Sigma_-(x, y))$. The Jacobian determinant J_- vanishes exactly where χ_- vanishes; this is the locus of points where reversing the orientation does not change the element in the moduli space. We present two viewpoints - one with the cusp point to the left and one to the right giving the “outside” and “inside” of a surface which appears smooth in \mathbb{R}^3 except at the cusp point.

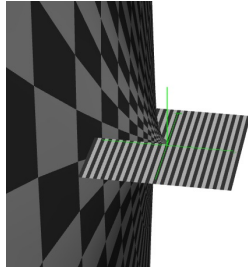


Fig. 4.5↑

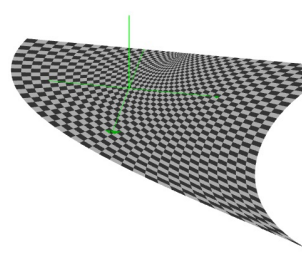


Fig. 4.6↑

Suppose ρ is indefinite. If $xy = 1$, we can change the parametrization so $x = y = 1$ and obtain a point of σ_+ . Otherwise $\chi_{0,1}(x, y) := \chi(\Gamma_{0,1}(x, y))$ vanishes precisely when $x = y$ and this is the range of σ_+ which is the right boundary of $\Theta_0(\mathfrak{Z}_0)$. Similarly $\chi_{0,2}(x, y) := \chi(\Gamma_{0,2}(x, y))$ vanishes precisely when $x = -y$ and this is the range of σ_- which is the left boundary of $\Theta_0(\mathfrak{Z}_0)$. Thus once again χ suffices to distinguish the relevant orientations. We omit pictures as they were not particularly useful.

5. TYPE \mathcal{B} STRUCTURES

Let $C \in \mathcal{Z}_{23\mathcal{B}}$ define a homogeneous structure on $\mathbb{R}^+ \times \mathbb{R}$ with $2 \leq \kappa(\mathcal{M}_C) \leq 3$. Use $\pm dx^1 \wedge dx^2$ to define a corresponding oriented surface \mathcal{M}_C^\pm . Let $\mathfrak{Z}_{23\mathcal{B}}^+$ be the associated moduli space of oriented surfaces with $2 \leq \kappa(\mathcal{M}_C^\pm) \leq 3$. Let

$$\begin{aligned}\mathfrak{G} &:= \{T : T(x^1, x^2) = (ax^1, bx^1 + cx^2 + d) \text{ for } a > 0 \text{ and } c \neq 0\}, \\ \mathfrak{J} &:= \{T_{b,c} \in \mathfrak{G} : T_{b,c}(x^1, x^2) = (x^1, bx^1 + cx^2) \text{ for } c \neq 0\}, \\ \mathfrak{J}^+ &:= \{T_{b,c} \in \mathfrak{J} \text{ for } c > 0\}.\end{aligned}$$

Our previous discussion in [2] permits us to identify

$$\mathfrak{Z}_{23\mathcal{B}} = \mathcal{Z}_{23\mathcal{B}}/\mathfrak{J} \text{ and } \mathfrak{Z}_{23\mathcal{B}}^+ = \mathcal{Z}_{23\mathcal{B}}/\mathfrak{J}^+.$$

This is a non-trivial assertion if $\kappa(\mathcal{M}) = 3$ as there are non-linear affine transformations. However, they play no role in defining the affine isomorphism type. In Lemma 5.1, we will define several invariant tensors on $\mathfrak{Z}_{23\mathcal{B}}$ that will play a crucial role in our subsequent discussion. In Lemma 5.2, we will examine the isotropy subgroup of the natural action of \mathfrak{J}^+ on $\mathcal{Z}_{23\mathcal{B}}$, we will show that $\mathfrak{Z}_{23\mathcal{B}}$ is a 4-dimensional real analytic manifold, and we will prove the natural projection π^+ from $\mathcal{Z}_{23\mathcal{B}}$ to $\mathfrak{Z}_{23\mathcal{B}}^+$ is a real-analytic \mathfrak{J}^+ fiber bundle. In Lemma 5.4, we show that $\mathfrak{Z}_{23\mathcal{B}}^+$ is simply connected and that the second Betti number is 2. In Theorem 5.5 we complete the proof of Theorem 1.8 by giving $\mathfrak{Z}_{23\mathcal{B}}$ a real analytic structure, by showing that the projection from $\mathfrak{Z}_{23\mathcal{B}}^+$ to $\mathfrak{Z}_{23\mathcal{B}}$ is a \mathbb{Z}_2 branched cover where the ramification set is a real analytic sub-manifold of co-dimension 2, and by demonstrating that $\mathfrak{Z}_{23\mathcal{B}}^+$ is simply connected and has second Betti number equal to 1.

Lemma 5.1. *The following tensors are invariantly defined on $\mathfrak{Z}_{23\mathcal{B}}$:*

$$\begin{aligned}\rho_1 &:= \frac{1}{x^1} \{ \Gamma_{12}^2 dx^1 \otimes dx^1 + \Gamma_{22}^2 dx^1 \otimes dx^2 - \Gamma_{12}^1 dx^2 \otimes dx^1 - \Gamma_{22}^1 dx^2 \otimes dx^2 \}, \\ \rho_2 &:= \Gamma_{ij}^k \Gamma_{kl}^l dx^i \otimes dx^j, \quad \rho_3 := \Gamma_{ik}^l \Gamma_{jl}^k dx^i \otimes dx^j, \quad \rho_0 := \Gamma_{ij}^j dx^i, \\ \rho_4 &:= \Gamma_{ij}^k C_{ak}^j C_{bc}^i dx^a \otimes dx^b \otimes dx^c.\end{aligned}$$

Proof. Since contracting an upper index against a lower index is an affine invariant, ρ_i for $i \neq 1$ is an affine invariant and hence invariant under \mathfrak{G} . Since we may express $\rho = \rho_1 + \rho_2 - \rho_3$, we may conclude ρ_1 is invariant under the action of \mathfrak{G} although not under the action of $\text{GL}(2, \mathbb{R})$. Since $\mathfrak{Z}_{23\mathcal{B}} = \mathcal{Z}_{23\mathcal{B}}/\mathfrak{J}$, the desired result follows. \square

Pull-back defines an action of \mathfrak{J} on \mathbb{R}^6 which may be described as follows:

$$\begin{aligned}(T_{b,c})_* \partial_{x^1} &= \partial_{x^1} + b \partial_{x^2}, \quad (T_{b,c})_* \partial_{x^2} = c \partial_{x^2}, \\ (T_{b,c}^* C)_{11}^1 &= C_{11}^1 + 2bC_{12}^1 + b^2C_{22}^1, \\ (T_{b,c}^* C)_{11}^2 &= \frac{C_{11}^2 + b(2C_{12}^2 - C_{11}^1) + b^2(C_{22}^2 - 2C_{12}^1) - b^3C_{22}^1}{c}, \\ (T_{b,c}^* C)_{12}^1 &= cC_{12}^1 + bcC_{22}^1, \\ (T_{b,c}^* C)_{12}^2 &= C_{12}^2 + bC_{22}^2 - b(C_{12}^1 + bC_{22}^1), \\ (T_{b,c}^* C)_{22}^1 &= c^2C_{22}^1, \text{ and } (T_{b,c}^* C)_{22}^2 = cC_{22}^2 - bcC_{22}^1.\end{aligned}$$

Let $\mathfrak{J}^C := \{T_{b,c} : T_{b,c}^* C = C\}$ and $\mathfrak{J}^{+,C} := \mathfrak{J}^C \cap \mathfrak{J}^+$ be the isotropy subgroups.

Lemma 5.2. *Adopt the notation established above.*

- (1) *If $\dim\{\mathfrak{K}(\mathcal{M}_C)\} = 4$, then $\mathfrak{J}^{+,C}$ is non-trivial.*
- (2) *If $\kappa(\mathcal{M}_C) \leq 3$, then $\mathfrak{J}^{+,C}$ is trivial and the following conditions are equivalent:*
 - (a) *\mathfrak{J}^C is non-trivial.*
 - (b) *$(C_{11}^2, C_{12}^1, C_{22}^2) = (0, 0, 0)$.*
 - (c) *\mathcal{M}_C is amphichiral, i.e. $\mathcal{M}_C^+ \approx \mathcal{M}_C^-$.*

- (3) $\mathfrak{Z}_{23\mathcal{B}}^+$ is a 4-dimensional real analytic manifold without boundary. The natural projection from $\mathcal{Z}_{23\mathcal{B}}$ to $\mathfrak{Z}_{23\mathcal{B}}^+$ is a real-analytic principal \mathfrak{J}^+ fiber bundle.

Proof. If $\kappa(\mathcal{M}_C) = 4$, then $C_{12}^1 = C_{22}^1 = C_{22}^2 = 0$ (see [2]). We have

$$\begin{aligned} (T_{b,c}^*C)_{12}^2 &= C_{12}^2, & (T_{b,c}^*C)_{11}^1 &= C_{11}^1, \\ (T_{b,c}^*C)_{11}^2 &= c^{-1}\{C_{11}^2 + 2bC_{12}^2 - bC_{11}^1\}. \end{aligned}$$

We distinguish cases:

- (1) If $C_{11}^2 = 0$ and if $2C_{12}^2 - C_{11}^1 = 0$, then $\mathfrak{J}^{+,C} = \mathfrak{J}^+$.
- (2) If $C_{11}^2 \neq 0$ and if $2C_{12}^2 - C_{11}^1 = 0$, then $\mathfrak{J}^{+,C} = \{T_{b,1}\}$.
- (3) If $2C_{12}^2 - C_{11}^1 \neq 0$, then $\mathfrak{J}^{+,C}$ is defined by $b = (c-1)C_{11}^2\{2C_{12}^2 - C_{11}^1\}^{-1}$.

For the remainder of the proof, we suppose that $2 \leq \kappa(\mathcal{M}_C) \leq 3$ or, equivalently, that $(C_{12}^1, C_{22}^1, C_{22}^2) \neq (0, 0, 0)$. We first show that $(T_{b,c}^*C) = C$ implies $c = 1$ and $b = 0$.

- (1) Suppose $C_{22}^1 \neq 0$. As $(T_{b,c}^*C)_{22}^1 = C_{22}^1$ and $c > 0$, we have $c = 1$. Setting $(T_{b,c}^*C)_{12}^1 = C_{12}^1$ then implies $b = 0$.
- (2) Suppose $C_{22}^1 = 0$ but $C_{12}^1 \neq 0$. As $(T_{b,c}^*C)_{12}^1 = C_{12}^1$, we have $c = 1$. Setting $(T_{b,c}^*C)_{11}^1 = C_{11}^1$ then yields $b = 0$.
- (3) Suppose $C_{22}^1 = 0$ and $C_{12}^1 = 0$ but $C_{22}^2 \neq 0$. As $(T_{b,c}^*C)_{22}^2 = C_{22}^2$, we have that $c = 1$. Setting $(T_{b,c}^*C)_{12}^2 = C_{12}^2$ then implies $b = 0$.

Suppose $T \in \mathfrak{J}^C$. Then $T^2 \in \mathfrak{J}^{+,C}$ and $T^2 = \text{id}$. Hence if $T \neq \text{id}$, then $T(x^1, x^2) = (x^1, -x^2)$. Thus \mathfrak{J}^C is non-trivial if and only if $T^*C = C$, i.e. $C_{11}^2 = C_{12}^1 = C_{22}^2 = 0$. This establishes the equivalence of Assertion 2-a and Assertion 2-b. The equivalence with Assertion 2-c is then immediate.

To prove Assertion 3, we must construct real analytic charts on $\mathfrak{Z}_{23\mathcal{B}}^+$ and show the natural projection π^+ from $\mathcal{Z}_{23\mathcal{B}}$ to $\mathfrak{Z}_{23\mathcal{B}}^+$ defines a real analytic principal fiber bundle. Let

$$\begin{aligned} \mathcal{O}_0^\pm &:= \{C \in \mathcal{Z}_{23\mathcal{B}} : \pm \rho_0(\partial_2) > 0\}, \\ \mathcal{O}_1^\pm &:= \{C \in \mathcal{Z}_{23\mathcal{B}} : \pm \rho_1(\partial_2, \partial_2) > 0\}. \end{aligned}$$

Since $(T_{b,c})_*(\partial_2) = c\partial_2$, these sets are open subsets of \mathbb{R}^6 which are invariant under the action of \mathfrak{J}^+ . If $\rho_0(C)(\partial_2) = 0$ and $\rho_1(C)(\partial_2, \partial_2) = 0$, then $C_{12}^1 + C_{22}^2 = 0$ and $C_{22}^1 = 0$ and thus $C_{12}^1 \neq 0$ since $C \in \mathcal{Z}_{23\mathcal{B}}$. Let

$$\begin{aligned} \mathcal{E} &:= \mathcal{Z}_{23\mathcal{B}} \cap \{\mathcal{O}_0^+ \cup \mathcal{O}_0^- \cup \mathcal{O}_1^+ \cup \mathcal{O}_1^-\}^c \\ &= \{C : C_{12}^1 + C_{22}^2 = C_{22}^1 = 0\} \cap \mathcal{Z}_{23\mathcal{B}} \\ &\subset \{C : C_{12}^1 \neq 0\} \end{aligned} \tag{5.a}$$

be the exceptional set. We will construct equivariant charts covering \mathcal{E} when we discuss Case 5.1.3 below; the description is a bit complicated so we postpone a precise definition until that time.

Case 5.1.1. Let $C \in \mathcal{O}_0^\pm$. We have $\rho_0(C)(\partial_{x^2}) \neq 0$. Let

$$b = b(C) := \mp \rho_0(C)(\partial_{x^1}) \text{ and } c = c(C) := |\rho_0(C)(\partial_{x^2})|^{-1}.$$

Let $\tilde{C} = \tilde{C}(C) := T_{b,1}^*T_{0,c}^*(C)$. We then have the relations

$$\rho_{0,\tilde{C}}(\partial_{x^1}) = 0 \text{ and } \rho_{0,\tilde{C}}(\partial_{x^2}) = \pm 1. \tag{5.b}$$

Furthermore, $T(C) := T_{b,0}T_{0,c} \in \mathfrak{J}^+$ is uniquely determined by the requirement that $\tilde{C} := T(C)^*(C)$ satisfies the relations of Equation (5.b), i.e. $\tilde{C}_{12}^1 + \tilde{C}_{22}^2 = \pm 1$

and $\tilde{C}_{11}^1 + \tilde{C}_{12}^2 = 0$. Introduce local coordinates by setting:

$$\begin{aligned}\tilde{C}_{11}^1 &= z_0^1, & \tilde{C}_{11}^2 &= z_0^2, & \tilde{C}_{12}^1 &= z_0^3, \\ \tilde{C}_{12}^2 &= -z_0^1, & \tilde{C}_{22}^1 &= z_0^4, & \tilde{C}_{22}^2 &= \pm 1 - z_0^3.\end{aligned}$$

This shows that $\mathcal{O}_0^\pm \rightarrow \pi^+(\mathcal{O}_0^\pm)$ is a \mathfrak{J}^+ principal bundle where $\mathfrak{U}_0^\pm := \pi^+(\mathcal{O}_0^\pm)$ can be identified with the open subset of \mathbb{R}^4 where $\rho_0(\tilde{C}(\vec{z}_0)) \neq 0$. Since $\tilde{C}_{12}^1 + \tilde{C}_{22}^2 \neq 0$, $2 \leq \kappa(\tilde{C}(\vec{z}_0)) \leq 3$.

Case 5.1.2. Let $C \in \mathcal{O}_1^\pm$. We apply the same argument as that given above to see there is a unique $T(C) \in \mathfrak{J}^+$ so that $\tilde{C} := T(C)^*C$ satisfies $\rho_{1,\tilde{C}}(\partial_{x^2}, \partial_{x^2}) = \pm 1$ and $\rho_{1,\tilde{C}}^s(\partial_{x^1}, \partial_{x^2}) = 0$, i.e. $\tilde{C}_{22}^1 = \mp 1$ and $\tilde{C}_{22}^2 - \tilde{C}_{12}^1 = 0$. We introduce local coordinates by setting:

$$\begin{aligned}\tilde{C}_{11}^1 &= z_1^1, & \tilde{C}_{11}^2 &= z_1^2, & \tilde{C}_{12}^1 &= z_1^3, \\ \tilde{C}_{12}^2 &= z_1^4, & \tilde{C}_{22}^1 &= \mp 1, & \tilde{C}_{22}^2 &= z_1^3.\end{aligned}$$

This shows that $\mathcal{O}_1^\pm \rightarrow \pi^+(\mathcal{O}_1^\pm)$ is a \mathfrak{J}^+ principal bundle where $\mathfrak{U}_1^\pm := \pi^+(\mathcal{O}_1^\pm)$ can be identified with the open subset of \mathbb{R}^4 where $\rho_1(\tilde{C}(\vec{z}_1)) \neq 0$. Since $\tilde{C}_{22}^1 \neq 0$, $2 \leq \kappa(\tilde{C}(\vec{z}_1)) \leq 3$.

Case 5.1.3. The final charts \mathcal{O}_3^\pm are a bit more complicated to define as ρ_3 is purely quadratic and does not contain any linear terms. We adopt the notation of Equation (5.a) and consider the exceptional set

$$\mathcal{E} := \{C : C_{12}^1 \neq 0, C_{22}^2 = -C_{12}^1, C_{22}^1 = 0\}.$$

We wish to fix the gauge. Let $\tilde{\mathcal{E}} := \{C \in \mathcal{E} : C_{12}^1 = \pm 1 \text{ and } C_{11}^1 = 0\}$. If $C \in \mathcal{E}$, let $b = b(C) := \mp \frac{1}{2}C_{11}^1$, let $c = c(C) := 1$, and let $\tilde{C} = T_{b,c}^*C$. The calculations performed in the proof of Lemma 5.1 then show $\tilde{C}_{11}^1 = 0$ and $\tilde{C}_{12}^1 = \text{sign}(C_{12}^1)$ or, equivalently, $\rho_3^{\tilde{C}}(\partial_{x^1}, \partial_{x^2}) = 0$ and $\rho_3^{\tilde{C}}(\partial_{x^2}, \partial_{x^2}) = 2(x^1)^{-2}$. Furthermore, (b, c) are uniquely specified by these equations. Thus $\mathcal{E} = \tilde{\mathcal{E}} \cdot \mathfrak{J}^+$. We use this as our ansatz; we wish to ensure $\rho_3(\partial_1, \partial_2) = 0$ and $C_{12}^1 = \pm 1$. Let $\vec{z}_3 := (z_3^1, z_3^2, z_3^3, z_3^4) \in \mathbb{R}^4$, define $\tilde{C}^\pm(\vec{z})$ by setting:

$$\begin{aligned}\tilde{C}_{11}^{\pm 2}(\vec{z}_3) &:= z_3^1, & \tilde{C}_{12}^{\pm 2}(\vec{z}_3) &:= z_3^2, & \tilde{C}_{22}^{\pm 1} &:= z_3^3, & \tilde{C}_{22}^{\pm 2} &:= z_3^4, & \tilde{C}_{12}^{\pm 1}(\vec{z}_3) &= \pm 1, \\ \tilde{C}_{11}^{\pm 1}(\vec{z}_3) &:= -\tilde{C}_{12}^{\pm 2}(\vec{z}_3) \mp \{\tilde{C}_{11}^{\pm 2}(\vec{z}_3)\tilde{C}_{22}^{\pm 1}(\vec{z}_3) + \tilde{C}_{12}^{\pm 2}(\vec{z}_3)\tilde{C}_{22}^{\pm 2}(\vec{z}_3)\}.\end{aligned}$$

Since $\tilde{C}_{12}^{\pm 1}(\vec{z}_3) = \pm 1$, $2 \leq \kappa(\tilde{C}^\pm(\vec{z}_3)) \leq 3$ and $\tilde{C}^\pm(\vec{z}_3) \in \mathcal{Z}_{23B}$. We compute

$$\begin{aligned}\rho_3(\partial_{x^1}, \partial_{x^2}) &= 0, \\ \rho_3(\partial_{x^2}, \partial_{x^2}) &= 1 + 2C_{12}^2C_{22}^1 + C_{22}^2C_{22}^2 = 1 + 2z_3^2z_3^3 + z_3^4z_3^4.\end{aligned}$$

Let $\tilde{\mathcal{O}}_3 := \{\vec{z}_3 : 1 + 2z_3^2z_3^3 + z_3^4z_3^4 > 0\}$. Then $\tilde{C}(\tilde{\mathcal{O}}_3)$ is a neighborhood of $\tilde{\mathcal{E}}$. Suppose

$$T_{b,c}^*\{\tilde{C}(\tilde{\mathcal{O}}_3)\} \cap \tilde{C}(\tilde{\mathcal{O}}_3) \neq \emptyset.$$

If $b \neq 0$, then $\rho_3((T_{b,c})^*\partial_{x^1}, (T_{b,c})^*\partial_{x^2}) \neq 0$. Consequently $b = 0$. To ensure $(T_{b,c}^*C)_{12}^1 = \pm 1$, we then need $c = 1$. Consequently exactly as in the special case that $C \in \mathcal{E}$, b and c are determined. This shows that

$$T_{b,c}^*\{\tilde{C}(\tilde{\mathcal{O}}_3)\} \cap \tilde{C}(\tilde{\mathcal{O}}_3) = \emptyset \text{ for } (b, c) \neq (0, 1). \quad (5.c)$$

Clear the previous notation and set $\Psi^\pm(\vec{z}_3, T_{b,c}) := T_{b,c}^*\tilde{C}^\pm(\vec{z}_3)$. The argument given above to establish Equation (5.c) shows that Ψ^\pm is a \mathfrak{J}^+ equivariant diffeomorphism from $\tilde{\mathcal{O}}_3 \times \mathfrak{J}^+$ to an open subset \mathcal{O}_3^\pm of \mathbb{R}^6 which contains \mathcal{E} . This provides the missing charts over which π is a principal bundle.

We have constructed coordinate charts $\{\mathfrak{U}_0^\pm, \mathfrak{U}_1^\pm, \mathfrak{U}_3^\pm\}$ on \mathfrak{Z}_{23B}^+ so that π^+ admits a section and thus the bundle is trivial over these charts. The construction is in

the real analytic category and the transition functions between coordinate charts are real analytic. This completes the proof of Assertion 3. \square

Remark 5.3. A special case of the affine structures C with $\kappa(\mathcal{M}_C) \leq 3$ and non-trivial \mathfrak{J}^C , as discussed in Lemma 5.2-(2), is given by

$$C_{11}^1 = C_{12}^2 = \pm C_{22}^1, \quad \text{and} \quad C_{11}^2 = C_{12}^1 = C_{22}^2 = 0.$$

These affine structures correspond to the only non-flat homogeneous affine surfaces that are metrizable and the associated pseudo-Riemannian metric is non-homogeneous [10].

Let

$$\begin{aligned} K &:= \{C \in \mathbb{R}^6 : (C_{12}^1, C_{22}^1, C_{22}^2) = (0, 0, 0)\}, \\ P^\pm &:= (C_{11}^1 = 1, C_{11}^2 = 0, C_{12}^1 = 0, C_{12}^2 = 0, C_{22}^1 = \pm 1, C_{22}^2 = 0), \\ K^\pm(b, c) &:= \left\{ \begin{array}{lll} C_{11}^1 = 1 \pm b^2, & C_{11}^2 = -bc^{-1}(1 \pm b^2), & C_{12}^1 = \pm bc, \\ C_{12}^2 = \mp b^2, & C_{22}^1 = \pm c^2, & C_{22}^2 = \mp bc \end{array} \right\}. \end{aligned}$$

Let $K^\pm := K^\pm(\mathbb{R} \times \mathbb{R}^+)$; a direct computation shows $K^\pm = P^\pm \cdot \mathfrak{J}^+$.

Lemma 5.4.

- (1) $\mathcal{Z}_{23\mathcal{B}} = \mathbb{R}^6 - K - K^+ - K^-$.
- (2) $\mathfrak{J}_{23\mathcal{B}}^+$ is simply connected and $\pi_2(\mathfrak{J}_{23\mathcal{B}}^+) = \mathbb{Z}$.
- (3) $H_{DeR}^1(\mathfrak{J}_{23\mathcal{B}}^+) = 0$ and $H_{DeR}^2(\mathfrak{J}_{23\mathcal{B}}^+) = \mathbb{R}$.

Proof. $C \in \mathcal{Z}_{23\mathcal{B}}$ if and only if $(C_{12}^1, C_{22}^1, C_{22}^2) \neq (0, 0, 0)$ and $\rho(C) \neq 0$. Suppose $(C_{12}^1, C_{22}^1, C_{22}^2) \neq (0, 0, 0)$ but $\rho(C) = 0$. Because $0 = \rho_{12} - \rho_{21} = (\Gamma_{12}^1 + \Gamma_{22}^2)/x^1$, we may set $\Gamma_{12}^1 = x$ and $\Gamma_{22}^2 = -x$. First suppose $C_{22}^1 \neq 0$ or, equivalently, $\rho_1(\partial_{x^2}, \partial_{x^2}) \neq 0$. We consider the shear $T : (x^1, x^2) \rightarrow (x^1, \varepsilon x^1 + x^2)$ which sends ∂_{x^1} to $\partial_{x^1} - \varepsilon \partial_{x^2}$ and ∂_{x^2} to ∂_{x^2} . By choosing ε appropriately, we can ensure that $\rho_1(\partial_{x^2}, \partial_{x^1}) = 0$ or, equivalently, that $\Gamma_{12}^1 = 0$ and set $x = 0$. We rescale x^2 to set $C_{22}^1 = \pm 1$. Setting $\rho_{12} = 0$ yields $C_{11}^2 = 0$. The remaining equations become

$$C_{12}^2(1 + C_{11}^1 - C_{12}^2) = 0 \quad \text{and} \quad (-1 + C_{11}^1 - C_{12}^2) = 0.$$

We conclude $C_{12}^2 = 0$ and $C_{11}^1 = 1$ so $C = P^\pm$; this gives rise to the surfaces K^\pm . Suppose next $C_{22}^1 = 0$. We have $\rho_{22} = -2x^2(x^1)^{-2}$ so $x = 0$. This implies $(C_{12}^1, C_{22}^1, C_{22}^2) = (0, 0, 0)$ contrary to our assumption. This establishes Assertion 1.

The principal bundle $\mathfrak{J}^+ \rightarrow \mathcal{Z}_{23\mathcal{B}} \rightarrow \mathfrak{J}_{23\mathcal{B}}^+$ gives rise to an associated long exact sequence of homotopy groups

$$\dots \rightarrow \pi_k(\mathfrak{J}^+) \rightarrow \pi_k(\mathcal{Z}_{23\mathcal{B}}) \rightarrow \pi_k(\mathfrak{J}_{23\mathcal{B}}^+) \rightarrow \pi_{k-1}(\mathfrak{J}^+) \dots \quad (5.d)$$

Since \mathfrak{J}^+ is contractible, the k^{th} homotopy group $\pi_k(\mathfrak{J}^+) = 0$ for all k and thus $\pi_k(\mathcal{Z}_{23\mathcal{B}})$ is isomorphic to $\pi_k(\mathfrak{J}_{23\mathcal{B}}^+)$ for any k . It therefore suffices to prove $\mathcal{Z}_{23\mathcal{B}}$ is simply connected and $\pi_2(\mathcal{Z}_{23\mathcal{B}}) = \mathbb{Z}$. A crucial point is that $\mathcal{Z}_{23\mathcal{B}}$ is obtained from \mathbb{R}^6 (which is contractible) by deleting the submanifold K (which has codimension 3) and the two surfaces K^\pm (which have codimension 4). Let P and Q be two points in $\mathcal{Z}_{23\mathcal{B}}$. Find a smooth curve γ in \mathbb{R}^6 from P to Q . Make γ transverse to K^\pm and to K . Then γ does not intersect these submanifolds and thus joins P to Q in $\mathcal{Z}_{23\mathcal{B}}$. Consequently $\mathcal{Z}_{23\mathcal{B}}$ is connected - i.e. $\pi_0(\mathcal{Z}_{23\mathcal{B}}) = 0$. Next let γ be a closed loop in $\mathcal{Z}_{23\mathcal{B}}$. Since \mathbb{R}^6 is contractible, we can construct a homotopy H from γ to the constant path in \mathbb{R}^6 . Make this homotopy transverse to K and to K^\pm . Again, for

dimensional reasons, the homotopy misses K^\pm and K . Consequently, $\pi_1(\mathcal{Z}_{23\mathcal{B}}) = 0$. Note that $\mathbb{R}^6 - K$ is homotopy equivalent to S^2 . Let $\gamma_0 : S^2 \rightarrow \mathbb{R}^6 - K$ generate

$$\pi_2(\mathbb{R}^6 - K) = \pi_2(S^2) = \mathbb{Z}.$$

Make γ_0 transverse to K^\pm ; again for dimensional reasons, γ_0 misses the surfaces K^\pm so we can regard $\gamma_0 \in \pi_2(\mathcal{Z}_{23\mathcal{B}})$. Since γ_0 generates $\pi_2(\mathbb{R}^6 - K) = \mathbb{Z}$, γ_0 generates a cyclic subgroup of $\pi_2(\mathcal{Z}_{23\mathcal{B}})$. Let $\gamma : S^2 \rightarrow \mathcal{Z}_{23\mathcal{B}}$. Construct a homotopy H between γ and $n \cdot \gamma_0$ in $\mathbb{R}^6 - K$. Make the homotopy (which is 3-dimensional) transverse to K^\pm (which are 2-dimensional). Thus H misses K^\pm and lives in $\mathcal{Z}_{23\mathcal{B}}$. This shows $\pi_2(\mathcal{Z}_{23\mathcal{B}}) = \mathbb{Z}$ and completes the proof of Assertion 2.

By the Hurewicz Theorem, $H_1(\mathfrak{Z}_{23\mathcal{B}}) = 0$ and $H_2(\mathfrak{Z}_{23\mathcal{B}}) = \mathbb{Z}$. The universal coefficient theorem shows dually $H^1(\mathfrak{Z}_{23\mathcal{B}}) = 0$ and $H^2(\mathfrak{Z}_{23\mathcal{B}}) = \mathbb{Z}$. We may use the de Rham theorem to identify topological cohomology with DeRham cohomology and verify Assertion 3. \square

Theorem 5.5. *Let $\mathfrak{Z}_{23\mathcal{B}} = \mathcal{Z}_{23\mathcal{B}}/\mathfrak{I} = \mathfrak{Z}_{23\mathcal{B}}^+/\{\mathfrak{I}/\mathfrak{I}^+\}$ be the moduli space of isomorphism classes of unoriented surfaces \mathcal{M}_C of Type \mathcal{B} with $2 \leq \kappa(\mathcal{M}_C) \leq 3$.*

- (1) $\mathfrak{Z}_{23\mathcal{B}}$ is a 4-dimensional real analytic manifold.
- (2) The projection π from $\mathcal{Z}_{23\mathcal{B}}$ to $\mathfrak{Z}_{23\mathcal{B}}$ is real analytic.
- (3) The projection σ from $\mathfrak{Z}_{23\mathcal{B}}^+$ to $\mathfrak{Z}_{23\mathcal{B}}$ is a real analytic 2-sheeted branched cover which is ramified over a real analytic manifold of dimension 2 which is isomorphic to $\mathbb{R}^2 - 0 \cup \mathbb{R}^2 - 0$.
- (4) $\mathfrak{Z}_{23\mathcal{B}}$ is simply connected and $H_{\text{DeR}}^2(\mathfrak{Z}_{23\mathcal{B}}) = \mathbb{R}$.

Proof. Let $T(x^1, x^2) = (x^1, -x^2)$. Then $[T]$ generates $\mathfrak{I}/\mathfrak{I}^+ = \mathbb{Z}_2$; we could have taken $T_{b,-1}$ for any $b \in \mathbb{R}$, but this is a particularly felicitous choice. Let \mathfrak{F} be the fixed point set of T in $\mathfrak{Z}_{23\mathcal{B}}^+$. We examine the action of T on the coordinate systems discussed previously to prove Assertion 1.

Case 5.2.1. The coordinate system \mathcal{O}_0^\pm . We have

$$\begin{aligned} (C^\pm)_{11}^1 &= z_0^{1,\pm}, & (C^\pm)_{11}^2 &= z_0^{2,\pm}, & (C^\pm)_{12}^1 &= z_0^{3,\pm}, \\ (C^\pm)_{12}^2 &= -z_0^{1,\pm}, & (C^\pm)_{22}^1 &= z_0^{4,\pm}, & (C^\pm)_{22}^2 &= \pm 1 - z_0^{3,\pm}, \\ T(C^\pm(z_0^{1,\pm}, z_0^{2,\pm}, z_0^{3,\pm}, z_0^{4,\pm})) &= C^\mp(z_0^{1,\mp}, -z_0^{2,\mp}, -z_0^{3,\mp}, z_0^{4,\mp}). \end{aligned}$$

Thus, since $T^*\rho_0(\partial_{x^2}) = -\rho_0(\partial_{x^2})$ and ρ_0 is non-trivial, $[T]$ acts without fixed points and preserves $\mathcal{O}_0^+ \cup \mathcal{O}_0^-$; it is a linear interchange of \mathcal{O}_0^+ and \mathcal{O}_0^- . Thus the quotient gives a coordinate chart \mathcal{O}_0 on $\mathfrak{Z}_{23\mathcal{B}}$ in such a way that σ is an unramified real analytic double covering projection.

Case 5.2.2. The coordinate system \mathcal{O}_3^\pm . We have

$$\begin{aligned} (C^\pm)_{11}^1 &= -z_3^{2,\pm} \mp \{z_3^{1,\pm} z_3^{3,\pm} + z_3^{2,\pm} z_3^{4,\pm}\}, \\ (C^\pm)_{12}^1 &= \pm 1, \\ (C^\pm)_{11}^2 &= z_3^{1,\pm}, & (C^\pm)_{12}^2 &= z_3^{2,\pm}, & (C^\pm)_{22}^1 &= z_3^{3,\pm}, & (C^\pm)_{22}^2 &= z_3^{4,\pm}, \\ T(C^\pm(z_3^{1,\pm}, z_3^{2,\pm}, z_3^{3,\pm}, z_3^{4,\pm})) &= (C^\mp)(-z_3^{1,\mp}, z_3^{2,\mp}, z_3^{3,\mp}, -z_3^{4,\mp}). \end{aligned}$$

Thus $[T]$ acts without fixed points and preserves $\mathcal{O}_3^+ \cup \mathcal{O}_3^-$; it is a linear interchange of \mathcal{O}_3^+ and \mathcal{O}_3^- since $T^*(C_{12}^1) = -(C_{12}^1)$. Thus the quotient gives a coordinate chart \mathcal{O}_3 on $\mathfrak{Z}_{23\mathcal{B}}$ in such a way that σ is an unramified real analytic double covering projection.

Case 5.2.3. The coordinate system \mathcal{O}_1^\pm . Here the analysis is less trivial. We have

$$\begin{aligned} (C_\pm)_{11}^1 &= z_1^{1,\pm}, & (C_\pm)_{11}^2 &= z_1^{2,\pm}, & (C_\pm)_{12}^1 &= z_1^{3,\pm}, \\ (C_\pm)_{12}^2 &= z_1^4, & (C_\pm)_{22}^1 &= \pm 1, & (C_\pm)_{22}^2 &= z_1^{3,\pm}. \\ T(C(z_1^{1,\pm}, z_1^{2,\pm}, z_1^{3,\pm}, z_1^{4,\pm})) &= C^\pm(z_1^{1,\pm}, -z_1^{2,\pm}, -z_1^{3,\pm}, z_1^{4,\pm}). \end{aligned}$$

Note that the exceptional points $P^\pm \in \mathfrak{F}^\pm$ and that K does not meet \mathcal{O}_1^\pm . In contrast to the previous two cases, T preserves \mathcal{O}_1^\pm separately and has non-trivial fixed point sets $\mathfrak{F}^\pm = \{\bar{z}_{1,\pm} : z_1^{2,\pm} = z_1^{3,\pm} = 0\}$. We introduce complex coordinates setting

$$w_1^{1,\pm} := z_1^{1,\pm} + \sqrt{-1}z_1^{4,\pm} \text{ and } w_1^{2,\pm} := z_1^{2,\pm} + \sqrt{-1}z_1^{3,\pm}.$$

Consider the map $\Phi^\pm(w_1^{1,\pm}, w_1^{2,\pm}) := (w_1^{1,\pm}, (w_1^{2,\pm})^2)$. We have $\mathfrak{F}^\pm = \mathbb{C} \times \{0\}$ and $\Phi^\pm(w_1^{1,\pm}, w_1^{2,\pm}) = \Phi^\pm(\tilde{w}_1^{1,\pm}, \tilde{w}_1^{2,\pm})$ if and only if $\sigma(w_1^{1,\pm}, w_1^{2,\pm}) = \sigma(\tilde{w}_1^{1,\pm}, \tilde{w}_1^{2,\pm})$. Thus Φ extends to a 1-1 map from $\mathbb{C}^2/\mathbb{Z}_2$ to \mathbb{C}^2 which gives coordinates on $\mathfrak{Z}_{23\mathcal{B}}$. This gives $\mathfrak{Z}_{23\mathcal{B}}$ the structure of a real-analytic 4-dimensional manifold in such a way that σ is a real analytic double covering which is ramified along $\mathfrak{F}^\pm = \mathbb{C} \times \{0\}$. Assertion 1 now follows.

Since σ is surjective and $\mathfrak{Z}_{23\mathcal{B}}^+$ is connected, we conclude $\mathfrak{Z}_{23\mathcal{B}}$ is connected. We give $\mathfrak{Z}_{23\mathcal{B}}$ a simplicial structure so that $[T]$ is a simplicial map and so that $[T]$ is 1-1 on each simplex. This implies that every simplicial curve α in $\mathfrak{Z}_{23\mathcal{B}}$ can be lifted to a curve in $\tilde{\alpha}$ in $\mathfrak{Z}_{23\mathcal{B}}^+$. The lift will in general not be unique even if the initial vertex is fixed if one of the subsequent vertices is in \mathfrak{F}^\pm . Choose the base point to be in the ramifying set. Let α be a closed curve in $\mathfrak{Z}_{23\mathcal{B}}$ starting and ending in the ramifying set. Lift α to a (possibly) not closed curve in $\mathfrak{Z}_{23\mathcal{B}}^+$. Since σ is 1-1 on \mathfrak{F}^{-1} , $\tilde{\alpha}$ is in fact a closed curve in $\mathfrak{Z}_{23\mathcal{B}}^+$. Since $\mathfrak{Z}_{23\mathcal{B}}^+$ is simply connected, $\tilde{\alpha}$ is homotopic to the constant curve at the base point. Applying σ then gives a homotopy of α in $\mathfrak{Z}_{23\mathcal{B}}$ to the constant path and shows $\mathfrak{Z}_{23\mathcal{B}}$ is simply connected. This proves Assertion 2.

Let g be the generator of $\pi_2(\mathcal{Z}_{23\mathcal{B}}) = \mathbb{Z}$. Let i be the inclusion of $\mathcal{Z}_{23\mathcal{B}}$ into $\mathbb{R}^6 - K = \mathbb{R}^3 \times (\mathbb{R}^3 - 0)$ and let π be the projection of $\mathbb{R}^3 \times (\mathbb{R}^3 - 0)$ onto $\mathbb{R}^3 - 0$. We have a commutative diagram defined by $\pi_* \circ i_*$ in homotopy

$$\begin{array}{ccc} \mathbb{Z} = \pi_2(\mathcal{Z}_{23\mathcal{B}}) & \xrightarrow{\sim} & \mathbb{Z} = \pi_2(\mathbb{R}^3 - 0) \\ T_* \downarrow & \circ & T_* \downarrow \\ \mathbb{Z} = \pi_2(\mathcal{Z}_{23\mathcal{B}}) & \xrightarrow{\sim} & \mathbb{Z} = \pi_2(\mathbb{R}^3 - 0) \end{array}$$

We have $T(C_{12}^1, C_{22}^1, C_{22}^2) = (-C_{12}^1, C_{22}^1, -C_{22}^2)$. Since T is the composition of 2 reflections of $\mathbb{R}^3 - 0$, T is homotopic to the identity and thus T_* is the identity on $\pi_2(\mathbb{R}^3 - 0)$. Consequently T_* is the identity on $\pi_2(\mathcal{Z}_{23\mathcal{B}})$. The long exact sequence of the fibration given in Equation (5.d) gives rise to a commutative diagram

$$\begin{array}{ccc} \mathbb{Z} = \pi_2(\mathcal{Z}_{23\mathcal{B}}) & \xrightarrow{\sim} & \mathbb{Z} = \pi_2(\mathfrak{Z}_{23\mathcal{B}})^+ \\ T_* \downarrow & \circ & [T]_* \downarrow \\ \mathbb{Z} = \pi_2(\mathcal{Z}_{23\mathcal{B}}) & \xrightarrow{\sim} & \mathbb{Z} = \pi_2(\mathfrak{Z}_{23\mathcal{B}})^+. \end{array}$$

The Hurewicz isomorphism then shows $[T]_* = \text{id}$ on $H_2(\mathfrak{Z}_{23\mathcal{B}}^+)$.

Let Δ be a simplex in $\mathfrak{Z}_{23\mathcal{B}}$. There are two possible lifts of Δ to $\mathfrak{Z}_{23\mathcal{B}}^+$; if Δ lies totally in \mathfrak{F} , we take 2 copies of the canonical lift. We define $\sigma^*(\Delta)$ to be the formal sum of these two lifts. This defines a chain map which induces a map in homology denoted by σ^* which satisfies

$$\sigma_* \sigma^* = 2 \text{ and } \sigma^* \sigma_* = \text{id} + [T]_*.$$

Since $[T]_*$ is the identity, we also obtain $\sigma^*\sigma_*$ is multiplication by 2. Consequently σ_* is an isomorphism from $H_2(\mathfrak{Z}_{23\mathcal{B}}^+, \mathbb{R})$ to $H_2(\mathfrak{Z}_{23\mathcal{B}}, \mathbb{R})$. Assertion 3 now follows via duality. \square

Remark 5.6. If σ was a covering projection, the proof of Assertion 3 that we have given is just the usual application of transfer and induction in homology; this is, for example, how the homology of real projective space with coefficients in \mathbb{R} is computed and exploits the fact that we are dealing with a Mackey functor. There is an additional subtlety here since the covering projection is in fact ramified that needed to be dealt with. Note that we obtain no information concerning any 2-torsion in $H_2(\mathfrak{Z}_{23\mathcal{B}}; \mathbb{Z})$ or, equivalently, in $\pi_2(\mathfrak{Z}_{23\mathcal{B}}; \mathbb{Z})$.

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